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Operator-theoretic methods in homogenisation of singular periodic structures

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The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Serena D'Onofrio

Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, with the exception of Chapter 2 and Chapters 4 to 6, which contain research articles that originated from collaboration with my supervisor Kirill Cherednichenko.

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Serena D'Onofrio

Contents

Introduction

Historical overview

Understanding how material behaviour emerges from the properties of constituent elements of a heterogeneous medium, modelled using a discrete or continuous approach (or a combination of the two), is a common theme in Mechanics, Physics, Engineering, Chemistry, Biology and many others fields of knowledge encompassed by the term "multiscale phenomena". In this context, the set of heuristic and rigorous approaches that allow one to obtain macroscopic models from microscopic ones across length-scales is often referred to as multiscale analysis. From the mathematical point of view, a convenient way to obtain information about this microscopic-to-macroscopic transition is to treat the ratio ε between the length-scales that characterise the different descriptions as an asymptotically small parameter (i.e. $\varepsilon \to 0$), corresponding to the so-called "homogenisation limit".

In the setting of continuous media with multiple scales, homogenisation theory deals with the analysis of partial differential equations with coefficients that oscillate with period ε. Its aim can therefore be formulated as the derivation of effective equations that describe the approximate behaviour "in the large" (i.e. the "macroscopic" behaviour) of the original "inhomogeneous" problem, by averaging out the small-scale oscillations. In order to make this process quantitative, one usually chooses a topology in the solution space and expresses the above behaviour in terms of this topology, possibly with convergence error estimates (when the topology is metrisable).

We illustrate the homogenisation approach with the following example. Let us consider an elliptic equation in the domain $\Omega \subseteq \mathbb{R}^d$ with ε-periodic coefficient $A = A(x/\varepsilon)$, $\varepsilon \ll 1$, where A is a Q-periodic, positive-definite, matrix-valued function ($Q = [0,1)^d$):

$$-\operatorname{div} A(x/\varepsilon)\nabla u^\varepsilon(x) + u^\varepsilon(x) = f(x), \qquad x \in \Omega, \tag{1}$$

for a given function f in an appropriate function space H (e.g., $H = L^2(\Omega)$) subject to some boundary conditions on $u^\varepsilon(x)$, $x \in \partial\Omega$. If one understands (1) in the weak sense, that is

$$u^\varepsilon \in H_{\rm b}, \qquad \int_\Omega A(x/\varepsilon)\nabla u^\varepsilon \cdot \nabla \phi \, dx + \int_\Omega u^\varepsilon \phi \, dx = \int_\Omega f\phi \, dx \qquad \forall \phi \in H_{\rm b},$$

for an appropriate linear dense subset $H_{\mathrm{b}} \subset H$, then we can associate, with the differential expression $-\operatorname{div}(A(x/\varepsilon)\nabla)$, a linear operator $\mathcal{A}^\varepsilon$ with domain $\operatorname{dom}(\mathcal{A}^\varepsilon) \subset H_{\mathrm{b}}$, which reflects the choice of the boundary conditions. The right-hand side f describes the density of the applied forces and the coefficients matrix A stands for the (microscopic) material properties of the medium. We refer to $x \in \Omega$ and $y \doteq x/\varepsilon \in Q$ as the slow and fast variable, respectively.

The standard elliptic theory (see e.g. the book of Zhikov, Kozlov and Oleinik [74]) yields a solution u^ε to (1) and proves, with different approaches, the convergence when $\varepsilon \to 0$ to u_0 in an appropriate function space (such as H above). The function u_0 solves

$$-\operatorname{div} A^{\mathrm{hom}}\nabla u_0 + u_0 = f, \qquad u_0 \in \operatorname{dom}(\mathcal{A}^{\mathrm{hom}}). \tag{2}$$

This is the so-called homogenised problem, where A^{hom} is the constant matrix representing the "effective" material properties of the homogenised medium. We associate to the differential expression $-\operatorname{div} A^{\mathrm{hom}}\nabla$ the homogenised operator $\mathcal{A}^{\mathrm{hom}}$ with domain $\operatorname{dom}(\mathcal{A}^{\mathrm{hom}})$, which describes the effective boundary conditions.

The present thesis is devoted to developing methods in homogenisation for a scalar elliptic operator and for the Maxwell operator of electromagnetism in the setting of arbitrary Borel measures. We study the convergence of the solution of our problems to the solution of the homogenised problems with different techniques, and we develop an original approach to achieve operator-norm resolvent estimates. Here we give a short historical overview of the literature and the papers which influenced and motivated the results of this work. This is not a comprehensive review of the homogenisation theory, but a collection of the more relevant papers that allow the development of this thesis.

Homogenisation theory has been extended in the last fifty years or so in different directions. It is difficult to pinpoint the exact beginning of this subject, but the first mathematical homogenisation approaches appear around 1960–1970. In this period we have the pioneering works of Marchenko and Hruslov [43], [44], Bogoliubov and Mitropolsky [15], and De Giorgi and Spagnolo [57].

Following these earlier works, homogenisation theory has been deeply studied and developed. We mention here the work [68] by Tartar, who proved a homogenisation theorem with the compensated compactness theory. At the same time De Giorgi and Franzoni in [30] developed the Γ-convergence technique to study the homogenisation of nonlinear problems. Meanwhile, Murat and Tartar in [50] introduced the concept of H-convergence to obtain homogenisation for linear elliptic equations with coefficients that are not necessarily symmetric.

One of the classical approaches that has motivated part of the results of this thesis is the method of multiple-scale asymptotic expansions. The technique of asymptotic expansions has been used for the construction of homogenised equations by Sanchez–Palencia [52] and Bakhvalov [3], [4]. Sanchez–Palencia obtained formal asymptotics, Bakhvalov proved that u_0 is the limit of u^ε, and provided corresponding error estimates. Subsequently, similar results have been obtained by Bensoussan, Lions and Papanicolaou

[6] and Keller [39]. Applications and extensions of the multi-scale expansions method in homogenisation theory, were summarised and developed in Sanchez-Palencia [53], Bakhvalov and Panasenko [2].

The idea of the multiple-scale expansions approach is to write the solution to a differential equation (e.g., (1)), in the form

$$u^\varepsilon(x) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n\left(x, \frac{x}{\varepsilon}\right), \tag{3}$$

where $u_n(x, y)$, $n = 0, 1, 2, \ldots$, is assumed to be periodic in the second variable $y \in Q$. Plugging the expansion (3) in the original equation and gathering terms with the same power of ε, one obtains a recurrence sequence of equations for each term $u_n(x, y)$. It follows that the first term in (3) is a function of x only, thus $u_0(x, y) = u_0(x)$. The corrector $u_1(x, y)$ has the form

$$u_1(x, y) = \sum_{k=1}^{d} N_k(y)\partial_{x_k} u_0(x),$$

where N_k is the zero-mean periodic solution of the 'cell problem'

$$\operatorname{div}_y A(y)\nabla_y N_k(y) = -\sum_{j=1}^{d} \partial_{y_j} A_{jk}(y),$$

where A_{ij} are the entries of the matrix A. The solvability condition for $u_2(x, y)$ leads to the homogenised equation for $u_0(x)$ of the form (2), where A^{hom} is the constant matrix given by

$$A^{\mathrm{hom}} = \int_Q A(y)(\nabla_y N(y) + I)\, dy,$$

where the entries of the matrix $\nabla_y N$ are given by $(\nabla_y N)_{ij} = N_{j,i}$. The multiple-scale approach consists of two steps: formally constructing the homogenised equation and proving the convergence of u^ε to u_0.

The above strategy has been used in a variety of multiscale contexts, going well beyond the scalar elliptic equation (1). Implementing the formal part of the method has usually been supported by convergence results (which still vary in terms of the choice of convergence topology and the rate of convergence shown). This has led to a general belief that expansions of the form (3) will provide the "correct" homogenised model, written as a problem for u_0 (e.g., given by (2) in the case of the original problem (1)), which will inevitably be substantiated by appropriate convergence statements. The role of the second, analytical, part of this process appears secondary, especially if one disregards its impact on the choice of numerical scheme. One of the outcomes of the present thesis consists in showing that, in the context of homogenisation for the Maxwell system for electromagnetism, a formal approach may simply provide an incorrect equation for u_0, which fully reinstates the role of rigorous analysis in the derivation of the homogenised model.

Another technique that has influenced our work is the two-scale convergence developed around 1990. The idea of the two-scale convergence is to preserve in the limit the information about oscillations of the elements of a function sequence on ε-scale. An advantage of this method is that it is self-contained: in a single process, it is possible to construct the homogenised equation and to prove the convergence. The two-scale convergence approach was introduced by Nguetseng in [51] in an abstract functional analytic form, and then Allaire in [1] developed further the theory and its applications to problems in homogenisation. One decade later, the method of two-scale convergence was generalised by Zhikov, see [71], [72] and [73], who extended it to the case of spaces with arbitrary periodic Borel measures, and applied it to the analysis of the double-porosity model and to elasticity problems for thin networks. At the same time, Bouchitté and Fragalà in [16] studied the homogenisation of thin structures with the method of two-scale convergence.

The above mentioned approaches do not suffice to address many questions important for physics applications, especially those of a qualitative nature, e.g., when the convergence of spectra can be guaranteed. In terms of dealing with these aspects, the operator-norm resolvent convergence (see, e.g., the book [38] by Kato) is an appropriate alternative. The first results about operator-norm resolvent convergence in homogenisation for "classical" problems are in the works of Sevostyanova [55] and Zhikov [70], where the asymptotic analysis of the Green functions of the corresponding problems is carried out. Following that, norm resolvent estimates were developed in the contest of self-adjoint elliptic operators as well as corresponding parabolic problems. In [75] and [76], Zhikov and Pastukova used the so-called method of first-order approximation, and in [32] and [33] Griso obtained norm-resolvent estimates with the method of periodic unfolding. Kenig, Lin and Shen in [40] and [41] adapted the classical boundary-layer potential analysis to the treatment of systems of PDE. Meanwhile, Birman and Suslina, with a method introduced in [8] and [11], proved sharp operator-norm estimates for self-adjoint operators of the form $X_t^* X_t$, where X_t is a linear operator pencil. Their approach is based on spectral perturbation theory applied to a wide class of operators admitting such a factorisation. It has been subsequently used by Suslina and her students to prove operator-norm and energy estimates for several related classes of problems. The relevant works concern boundary-value operators [62], [63], parabolic semigroups [59], [61], [47], hyperbolic groups [12], [45], [46] and perforated domains [65]. Furthermore, the stationary system of Maxwell equations has been analysed in the whole space setting [8, Chapter 7], [10], [58], [60] and in a bounded domain [66], [67]. Recently, the case of non-local elliptic operators was analysed by Senik in [54].

Object of the present thesis

The results about operator-norm convergence mentioned above, concern homogenisation problems in the Lebesgue measure setting. Our interest is the analysis of spaces with arbitrary Borel measures, that is the study of particular geometric objects described by a wide class of measures, which often arise in applications.

The goal of this thesis is to provide asymptotic estimates for a scalar elliptic operator and for a vectorial problem for the Maxwell system for electromagnetism. In particular we aim to prove operator-norm convergence estimates, that is estimates where the approximation rate is proportional to the norm of the function representing the applied forces. We develop our analysis in the setting of singular periodic structures, which implies the presence of an arbitrary periodic Borel measure in the space.

To describe the singular structures, we start introducing the thin structures, which naturally arise in applications when one considers, for example, the propagation of waves in a network of thin domains. Thin structures are composite media whose properties on the scale of the unit cell Q involve a small geometric parameter δ that goes to zero together with the period ε. Singular structures are Q-periodic sets of lower dimension then the ambient space $\mathbb{R}^d$, which can be viewed as "limiting" thin structures by formally setting $\delta = 0$. The simplest singular periodic structure is a straight line segment Q-periodically extended to the whole plane. Other examples are a square net or a system of parallel wires. It is important to highlight that each singular object carries a natural measure μ which is singular with respect the Lebesgue measure. In general, a periodic singular object is described by a periodic Borel measure μ with periodicity cell $Q = (0,1]^d$ such that $\int_Q d\mu = 1$. The singular structure itself is the support of the measure.

To formulate the homogenisation problem, we introduce the measure μ^ε (see for example the works by Zhikov [71] and [72]) defined by

$$\mu^\varepsilon(B) = \varepsilon^d \mu(\varepsilon^{-1}B)$$

for all Borel sets $B \subset \mathbb{R}^d$. The measure μ^ε is ε-periodic, and we have

$$\int_{\varepsilon Q} d\mu^\varepsilon = \varepsilon^d \int_Q d\mu = \varepsilon^d.$$

It follows that the measure μ^ε weakly converges to the Lebesgue measure, that is

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \phi \, d\mu^\varepsilon = \int_{\mathbb{R}^d} \phi \, dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^d).$$

The first problem we analyse in the present thesis, is the following scalar elliptic problem on singular periodic structures:

$$-\operatorname{div} A(\cdot/\varepsilon)\nabla u^\varepsilon + u^\varepsilon = f^\varepsilon, \quad f^\varepsilon \in L^2(\mathbb{R}^d, d\mu^\varepsilon). \tag{4}$$

To understand how the presence of an arbitrary measure influences the problem and to highlight the differences with the problem (1), we introduce the space $H^1(\mathbb{R}^d, d\mu^\varepsilon)$. It is defined as the closure of the set $\{(\phi, \nabla\phi), \phi \in C_0^\infty(\mathbb{R}^d)\}$ in the norm of $L^2(\mathbb{R}^d, d\mu^\varepsilon) \oplus \left[L^2(\mathbb{R}^d, d\mu^\varepsilon)\right]^d$. We say that $(u^\varepsilon, \nabla u^\varepsilon) \in H^1(\mathbb{R}^d, d\mu^\varepsilon)$ solves (4) if

$$\int_{\mathbb{R}^d} A(\cdot/\varepsilon)\nabla u^\varepsilon \cdot \overline{\nabla\phi}\, d\mu^\varepsilon + \int_{\mathbb{R}^d} u^\varepsilon\overline{\phi}\, d\mu^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon\overline{\phi}\, d\mu^\varepsilon \quad \forall(\phi, \nabla\phi) \in H^1(\mathbb{R}^d, d\mu^\varepsilon).$$

It is important to remark (see Chapter 1) that for u^ε in $L^2(\mathbb{R}^d, d\mu^\varepsilon)$ there exists more than one gradient ∇u^ε. The Riesz representation theorem implies the existence and uniqueness of solutions regarded as a pair $(u^\varepsilon, \nabla u^\varepsilon)$. Starting from the problem (4), our purpose is to estimate uniformly u^ε with the solution of the following homogenised equation:

$$\operatorname{div} A^{\text{hom}}\nabla u^\varepsilon_{\text{hom}} + u^\varepsilon_{\text{hom}} = f^\varepsilon, \quad f^\varepsilon \in L^2(\mathbb{R}^d, d\mu^\varepsilon). \tag{5}$$

The second problem object of this thesis is the one for the system of Maxwell equations on singular periodic structures. The formulation we refer to in this work follows the books of Jackson [35] and Cessenat [17] (the precise construction to obtain the non-dimensional system can be found in the Introduction of Chapter 3). The problem we consider in our analysis is the following:

$$\begin{cases} \operatorname{curl} \widetilde{A}(\cdot/\varepsilon)B_\varepsilon - D_\varepsilon = g^\varepsilon, & g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \\ \operatorname{curl} A(\cdot/\varepsilon)D_\varepsilon + B_\varepsilon = f^\varepsilon, & f^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon). \end{cases} \tag{6}$$

Here B_ε is the magnetic induction, $H_\varepsilon := \widetilde{A}^{-1}B_\varepsilon$ the magnetic field, D_ε the electric displacement and $E_\varepsilon := A^{-1}D_\varepsilon$ the electric field. With A we denote the inverse of the relative electric permittivity, with $\widetilde{A}$ the inverse of the relative magnetic permeability. The function g^ε is the divergence-free representation of the external currents of the system, and f^ε is an auxiliary divergence-free function introduced in order to deal with the Maxwell operator.

The homogenised system related to (6) suggested by the formal approach, has the following form:

$$\begin{cases} \operatorname{curl} \widetilde{A}^{\text{hom}}B^{\text{hom}}_\varepsilon - D^{\text{hom}}_\varepsilon = g^\varepsilon, & g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \\ \operatorname{curl} A^{\text{hom}}D^{\text{hom}}_\varepsilon + B^{\text{hom}}_\varepsilon = f^\varepsilon, & f^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon). \end{cases} \tag{7}$$

Here $B^{\text{hom}}_\varepsilon$, $H^{\text{hom}}_\varepsilon := (\widetilde{A}^{\text{hom}})^{-1}B^{\text{hom}}_\varepsilon$, $D^{\text{hom}}_\varepsilon$ and $E^{\text{hom}}_\varepsilon := (A^{\text{hom}})^{-1}D^{\text{hom}}_\varepsilon$ are the homogenised fields and displacements.

Our goal is to obtain operator-norm estimates for the system (6). However, the solution of the system (7) is not operator-norm close to the solution of the original problem, even in the setting of Lebesgue measure (see the works by Birman and Suslina [10], [58] and [60]).

In order to deal with the system (6), we split it into different cases, which we analyse separately. For the first case, we set the relative magnetic permeability to unity and assume that the external currents, represented for each value ε by the function g^ε, vanish. Hence, for $\widetilde{A} = I$ and $g^\varepsilon = 0$ in the general system (6), we obtain the following problem:

$$\operatorname{curl} A(\cdot/\varepsilon) \operatorname{curl} B_\varepsilon + B_\varepsilon = f^\varepsilon, \quad f^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \operatorname{div} f^\varepsilon = 0. \tag{8}$$

Here the magnetic induction B_ε coincides with the magnetic field H_ε. The solutions of equation (8) are understood as pairs $(B_\varepsilon, \operatorname{curl} B_\varepsilon)$ in the space $H^1_{\text{curl}}(\mathbb{R}^3, d\mu^\varepsilon)$, defined as the closure of the set of pairs $\{(\phi, \operatorname{curl}\phi),\ \phi \in C_0^\infty(\mathbb{R}^3)\}$ in the direct sum $L^2(\mathbb{R}^3, d\mu^\varepsilon) \oplus L^2(\mathbb{R}^3, d\mu^\varepsilon)$. We say that $(B_\varepsilon, \operatorname{curl} B_\varepsilon) \in H^1_{\text{curl}}(\mathbb{R}^3, d\mu^\varepsilon)$ is a solution to (8) if

$$\int_{\mathbb{R}^3} A(\cdot/\varepsilon) \operatorname{curl} B_\varepsilon \cdot \overline{\operatorname{curl}\phi}\, d\mu^\varepsilon + \int_{\mathbb{R}^3} B_\varepsilon \cdot \overline{\phi}\, d\mu^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon \cdot \overline{\phi}\, d\mu^\varepsilon \quad \forall (\phi, \operatorname{curl}\phi) \in H^1_{\text{curl}}(\mathbb{R}^3, d\mu^\varepsilon).$$

In this case we estimate uniformly the difference between B_ε and $B_\varepsilon^{\text{hom}}$, where $B_\varepsilon^{\text{hom}}$ is the solution of the homogenised equation obtained setting $\widetilde{A}^{\text{hom}} = I$ and $g^\varepsilon = 0$ in (7). The result is different for the electric displacement and the electric field. In fact, the estimates for D_ε and E_ε contain respectively $D_\varepsilon^{\text{hom}}$ and $E_\varepsilon^{\text{hom}}$, solutions of the homogenised system obtained setting $\widetilde{A}^{\text{hom}} = I$ and $g^\varepsilon = 0$ in (7). Furthermore, in the leading-order terms of both estimates appears an additional term which rapidly oscillates when ε goes to zero.

For the second case, we set the relative magnetic permeability to unity, and assume that the external currents do not vanish. Hence, for $\widetilde{A} = I$ and $f^\varepsilon = 0$ in (6) we obtain the following problem written in terms of the electric displacement:

$$A^{1/2} \operatorname{curl}\operatorname{curl}(A^{1/2} D_\varepsilon) + D_\varepsilon = -A^{1/2} g^\varepsilon, \quad g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \operatorname{div} g^\varepsilon = 0. \tag{9}$$

The solutions of (9) are understood as pairs $(D_\varepsilon, \operatorname{curl}(A^{1/2} D_\varepsilon)$ in the Sobolev space $H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$, defined as the closure of the set of pairs $\{(\phi, \operatorname{curl}(A^{1/2}\phi)),\ \forall \phi \in C_0^1(\mathbb{R}^3)\}$ in the direct sum $L^2(\mathbb{R}^3, d\mu^\varepsilon) \oplus L^2(\mathbb{R}^3, d\mu^\varepsilon)$. We have that $(D_\varepsilon, \operatorname{curl}(A^{1/2} D_\varepsilon) \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$ is a solution of the problem (9) if

$$\int_{\mathbb{R}^3} \operatorname{curl}(A^{1/2} D_\varepsilon) \cdot \overline{\operatorname{curl}(A^{1/2}\phi)}\, d\mu^\varepsilon + \int_{\mathbb{R}^3} D_\varepsilon \cdot \overline{\phi}\, d\mu^\varepsilon = -\int_{\mathbb{R}^3} g^\varepsilon \cdot \overline{\phi}\, d\mu^\varepsilon$$
$$\forall (\phi, \operatorname{curl}(A^{1/2}\phi)) \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon).$$

In this case the operator-norm estimates are more complicated. In fact, D_ε is approximated by an expression depending on ε and $y \in Q$, represented by a pseudo-differential operator which is, in some sense, a singular perturbation of (7). In the Section "Structure of the thesis and results" we formally explain the structure of this expression, but we do not write it as the solution of a system similar to (6). Analogous results are obtained as well for the electric field and for the magnetic field, which coincides with the magnetic induction in this case.

For the third case, we consider the general Maxwell system (6), where the relative magnetic permeability is an arbitrary matrix-valued function. Without loss of generality we set $f^\varepsilon = 0$ in (6), and we obtain the following equation in terms of the electric displacement:

$$A^{1/2}\operatorname{curl}\widetilde{A}\operatorname{curl}(A^{1/2}D_\varepsilon) + D_\varepsilon = -A^{1/2}g^\varepsilon, \quad g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \operatorname{div} g^\varepsilon = 0. \tag{10}$$

The solution of such problem is understood as the pair $(D_\varepsilon, \operatorname{curl}(A^{1/2}D_\varepsilon))$ in the space $H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$, such that the following integral identity holds:

$$\int_{\mathbb{R}^3} \widetilde{A}\operatorname{curl}(A^{1/2}D_\varepsilon)\cdot\overline{\operatorname{curl}(A^{1/2}\phi)}\,d\mu^\varepsilon + \int_{\mathbb{R}^3} D_\varepsilon\cdot\overline{\phi}\,d\mu^\varepsilon = -\int_{\mathbb{R}^3} g^\varepsilon\cdot\overline{\phi}\,d\mu^\varepsilon$$
$$\forall(\phi, \operatorname{curl}(A^{1/2}\phi)) \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon).$$

The operator-norm estimates in this case have the same structure as the estimates developed for (9). In fact, as in the previous case, the solution of (10) is not operator-norm close to the solution of the formal limit system (7). Here as well D_ε is approximated by an expression depending on ε and $y \in Q$, represented by a pseudo-differential operator. The presence of $\widetilde{A}$ in (10) does not influence the structure of the operator.

Strategy for the present study

The results developed by Birman and Suslina have a motivational value for the following work. The abstract theoretical method developed by them in [8] and [11] allows the obtaining of norm resolvent estimates for a wide class of operators relevant from a physical point of view, as for example Maxwell equations system. Furthermore an important contribution is given by the work of Cherednichenko and Cooper [26] where norm resolvent estimates are provided for the high contrast elliptic problems using the Floquet-Gelfand transform setting. Their approach is based on the uniform power-series asymptotic analysis of the fibre operators in the associated direct integral. Our question is whether one can try to combine these ideas and build on a basis of them a method for arbitrary measures. In the present thesis we achieve this goal.

The following work is divided into two main parts: in Chapter 1, 2 we focus our analysis on the scalar problem for the elliptic operator, in Chapters 3–6 we study the vectorial problem for the system of Maxwell equations. In both parts, we start the analysis of the operator with two standard methods: the two-scale convergence and the two-scale asymptotic expansions (see Chapter 1, 3). Secondly, we describe our original approach, to obtain operator-norm resolvent estimates for the scalar and the vectorial problems.

The method we develop to tackle these problems starts with the definition of suitable

Sobolev spaces with respect to an arbitrary measure μ supported on the singular periodic structure itself. The idea arises from the works of Zhikov [71], [72], [73] where is defined the space $H^1_{\#}(Q, d\mu)$ of periodic functions H^1 in the unitary cell Q with respect to arbitrary measures. In particular, he introduced the idea of gradient with respect to a measure. The challenge at the initial level is to deal with gradients, divergences and curls depending on arbitrary measures, in fact they are not uniquely defined. Furthermore, it is important to understand how to define the solution of a problem with respect to an arbitrary measure.

To obtain operator-norm resolvent estimates we introduce a representation for functions in $L^2(\mathbb{R}^d, d\mu^\varepsilon)$ unitarily equivalent to the Gelfand transform, originally defined by Gelfand in [31] for the Lebesgue measure. The Gelfand transform for an arbitrary measure μ is discussed in detail by Zhikov and Pastukova in their work [77]. Here, we use its Floquet version to transform the original operator problem defined in the whole space $\mathbb{R}^d$ with a measure μ^ε, into a family of operators in the unitary cell Q with the measure μ. This family of operators is parametrised by the "quasimomentum" θ which in some sense replaces the macroscopic variable. We denote with Q' the dual cell $Q' = [-\pi, \pi)^d$, and we say that θ is in $\varepsilon^{-1}Q'$. The Floquet transform we use to construct the parametrised problem maps

$$L^2(\mathbb{R}^d, d\mu^\varepsilon) \to L^2(Q \times \varepsilon^{-1}Q', d\mu \times d\theta).$$

It is defined as follows:

$$(\mathcal{U}_\varepsilon u)(y, \theta) = \left(\frac{\varepsilon^2}{2\pi}\right)^{d/2} \sum_{n \in \mathbb{Z}^d} u(\varepsilon y + \varepsilon n) \exp(-i\varepsilon n \cdot \theta), \quad y \in Q,\ \theta \in \varepsilon^{-1}Q'. \tag{11}$$

The power of this machinery is that one passes from considering an operator on the unbounded domain $\mathbb{R}^d$, to considering a family of operators on the bounded periodic cell Q.

The core of our work is the analysis uniform in θ of the transformed resolvent operator. To obtain it we develop a special Poincaré-type inequality, which takes into account the quasiperiodicity of functions involved and the fact that the measure μ is arbitrary. The Poincaré-type inequality allows us to understand the geometry of the space, and it is the main challenge to achieve, especially in the problem for the system of Maxwell equations, in particular for the case with unitary relative magnetic permeability and non-zero current discussed in Chapter 5. The structure of a Poincaré-type inequality is the basis of our strategy, in fact it is useful to construct an asymptotic approximation in powers of ε for the solution. Then we carefully analyse the homogenisation corrector as a function of ε and θ, and estimate the remainder uniformly with respect to θ.

Our method is a reinterpretation and a generalisation of what Birman and Suslina did in their works. The main difference is that with our strategy we extract from the Poincaré-type inequality a uniform leading order term in the asymptotic approximation.

Birman and Suslina, with their approach, study the spectral germ for operator pencils, which quantifies the leading order of frequency dispersion of waves in a heterogeneous medium near the bottom of the spectrum of the associated differential operator with periodic coefficients.

The advantage of our technique arises in the analysis of the system of Maxwell equations, in particular in the non-magnetic case with non-zero external currents, and subsequently in the full Maxwell system.

Classical results in homogenisation for the Maxwell system, see for example the books by Bensoussan, Lions and Papanicolaou [6], by Bakhvalov and Panasenko [2], by Zhikov, Kozlov, Oleinik [74], and the work [69] by Wellander, only allow to obtain weak convergence to the solutions of the formal homogenised system. Operator-norm resolvent estimates have been obtained by Suslina in [60] for the full system of Maxwell equations, with the abstract method she developed with Birman in [8] and [11]. The innovation in Suslina's results is that the spectral analysis of the Maxwell operator allows the construction of a special corrector depending on ε, which enters the leading order term of the approximation, together with the solution of the formal homogenised equation. In our approach the challenge of constructing the corrector that will yield to operator-norm estimates, is contained within the task of the derivation of suitable Poincaré-type inequalities.

The method developed in this thesis can be applied to several problems studied in continuous mechanics (see e.g. the book [48] by Milton). Elasticity equation for plates, rods, shells, piezo-electricity equations, piezo-elasticity and much more. The analytical tools have to be modified and adapted to different settings, but the principal idea of our technique can be used. The case of elastic plates has been recently analysed by Cherednichenko and Velčić in [24], and the case of thin elastic rods is a work in progress by Cherednichenko, Velčić and Zubrinić [25].

Structure of the thesis and results

In Chapter 1 we start the analysis of the derivation of homogenised equation for the scalar problem of an elliptic operator in singular periodic structures. The main focus is to understand the properties of the homogenised equation in a Sobolev space with the arbitrary measures. In Section 1.1 we introduce the mathematical framework necessary to deal with partial differential equations on singular periodic structures. We define the Sobolev space of function H^1 with respect to arbitrary measures, the gradient and the divergence with respect to a measure μ. In particular we describe the link between the μ-gradient and the μ-divergence following the ideas of Zhikov in [73], in order to define in some sense the notion of "integration by parts" in spaces with measure. In Section 1.2 we start the analysis of the elliptic problem (4), and we study the limit equation for (4) with the method of two-scale convergence. In particular, we follow the idea of Kamotski and Smyshlyaev in their work [36], where they prove the two-scale convergence for high-contrast equations with periodic coefficients in the Lebesgue measure setting. We extend in a straightforward way their method to the case of singular periodic structures. The aim of Section 1.3 is to construct the homogenised equation for the elliptic problem (4) with the two-scale expansion technique described above. For this approach we only provide a formal expansion for u^ε.

In Chapter 2 we present the result of the work [22] by Cherednichenko and D'Onofrio. The analysis is focused on obtaining operator-norm resolvent estimate for u^ε solution of (4). In Section 2.1 we introduce the problem and we define the spaces $H^1_{\varepsilon\theta}(Q, d\mu)$ of quasiperiodic functions in H^1, with respect to a measure μ. Through the Floquet transform we obtain the following family of operator problem parametrised by the quasi-momentum θ:

$$-\varepsilon^{-2}\overline{e_{\varepsilon\theta}}\nabla \cdot A\nabla(e_{\varepsilon\theta}u^\varepsilon_\theta) + u^\varepsilon_\theta = F, \quad F \in L^2(Q, d\mu), \tag{12}$$

where $e_{\varepsilon\theta} := \exp(i\varepsilon\theta \cdot y)$.

Section 2.2 is dedicated to the formulation of the main result of this chapter, the operator-norm resolvent estimate for u^ε_θ solution of the transformed problem (12).

Theorem. *(See Theorem 2.2.3.) The following estimate holds with a constant $C > 0$ independent of ε, θ and F:*

$$\|u^\varepsilon_\theta - c_\theta\|_{L^2(Q,d\mu)} \leq \varepsilon C\|F\|_{L^2(Q,d\mu)},$$

where

$$c_\theta := \left(\theta \cdot A^{\hom}\theta + 1\right)^{-1} \int_Q F d\mu, \quad \theta \in \varepsilon^{-1}Q'.$$

Applying back the Floquet transform we prove the uniform error estimate in the whole space setting for u^ε solution of equation (4).

Corollary. *(See Corollary 2.2.4.) There exists a constant $C > 0$ independent of ε and f^ε such that*

$$\|u^\varepsilon - u^\varepsilon_{\hom}\|_{L^2(\mathbb{R}^d,d\mu^\varepsilon)} \leq \varepsilon C\|f^\varepsilon\|_{L^2(\mathbb{R}^d,d\mu^\varepsilon)},$$

where $u^\varepsilon_{\hom}$ is the solution of the homogenised equation (5)*.*

Hence, in the setting of singular periodic structure, we approximate uniformly the solution of the scalar elliptic problem u^ε with the solution of the formal homogenised equation. Section 2.3 and Section 2.4 are devoted to the proof of the estimates, based on the construction of an asymptotic approximation in power of ε for u^ε_θ. The main tool which allows us to obtain estimates uniform in θ for the reminder of such approximation is a Poincaré-type inequality. In fact, we prove this inequality for quasiperiodic functions with respect to an arbitrary Borel measures.

In the second part of the thesis we analyse the vectorial problem for the system of Maxwell equations in electromagnetism.

Chapter 3 is devoted to the introduction of the vectorial problem, and the analysis is focused on the homogenisation of harmonic in time system of Maxwell equations (6). In Section 3.1 we describe the mathematical framework necessary to study the Maxwell system in the setting of arbitrary measures. We define the Sobolev space of $H^1_{\mathrm{curl}}(\mathbb{R}^3, d\mu^\varepsilon)$ vectorial functions and we provide, in analogy with the definition of the gradient, a definition of curl with respect to a measure. In Section 3.2 we analyse the case of the non-magnetic system of Maxwell equations with zero external currents (that is problem (8)). In analogy with Section 1.2, here we adapt to the vectorial equation (8) the technique developed by Kamotski and Smyshlyaev in [36] based on the two-scale convergence. We construct the limit equation and we prove the two-scale convergence for B_ε.

In Section 3.3 we study the Maxwell system with relative magnetic permeability set to unity, through the two-scale asymptotic expansion approach. We split this problem into two cases: the one where the external currents vanish, that is equation (8), and the one with non-zero external currents, obtained setting $f^\varepsilon = 0$ and $\widetilde{A} = I$ in (6). For both cases we formally obtain that the leading order term in the asymptotic expansions is a constant vector, solution of the related homogenised equation. It is important to note that this result is not operator-norm close to the solution of the original problem in the case of non-zero external current, even in the setting of Lebesgue measure. In fact in the work [10] Birman and Suslina proved norm resolvent estimates for the Maxwell system with constant magnetic permeability in the whole-space with the Lebesgue measure. Their approximation term is not only the solution of the formal homogenised equation, in fact appears an additional special corrector depending on ε.

In Section 3.4 we analyse the full Maxwell system (6), and we formally construct the homogenised system with the method of two-scale asymptotic expansion. Here as well the limit system turns out to be an incorrect model if one were to require norm-resolvent (or even strong) convergence, as $\varepsilon \to 0$. This result was proved in the case of Lebesgue measure by Suslina in the works [58] and [60], where she obtained operator-norm resolvent estimates for the general Maxwell system. Her estimates highlight the presence of an additional special corrector depending on ε, which enters in the leading order term in the approximation.

In Chapter 4 we prove norm resolvent estimates for the system of Maxwell equations in the case with unitary magnetic permeability and zero external current (that is problem (8)). The content of this chapter can be found in the work of Cherednichenko and D'Onofrio [20]. The idea is to adapt the method developed in Chapter 2 for the scalar problem (see [22]), to the vectorial system of Maxwell equations. In Section 4.1 we introduce the problem and we describe the space $H^1_{\varepsilon\theta,\mathrm{curl}}(Q, d\mu)$ of quasiperiodic H^1_{curl} function with respect to the measure μ. Following the same idea developed for the scalar elliptic problem we obtain, with the Floquet transform, the unitary equivalent family of operator problem parametrised by the quasimomentum θ:

$$\varepsilon^{-2}\overline{e_{\varepsilon\theta}}\,\mathrm{curl}\,A\,\mathrm{curl}(e_{\varepsilon\theta}B^\varepsilon_\theta) + B^\varepsilon_\theta = F, \quad F \in L^2(Q, d\mu), \quad \overline{e_{\varepsilon\theta}}\,\mathrm{div}(e_{\varepsilon\theta}F) = 0. \tag{13}$$

Section 4.2 provides useful tools in the asymptotic analysis of (13). The first one is a special version of a Helmholtz decomposition for quasiperiodic functions, which allows us to represent the space as a sum of $\operatorname{div} e_{\varepsilon\theta}$-free and $\operatorname{curl} e_{\varepsilon\theta}$-free functions. The second tool is a Poincaré-type inequality with respect to arbitrary Borel measures, for quasiperiodic functions. In Section 4.3 we formulate the main result of the chapter, that is the uniform operator-norm resolvent estimate for B_θ^ε the solution of the problem (13).

Theorem. *(See Theorem 4.4.2.) The following estimate holds with a constant $C > 0$ independent of ε, θ and F:*

$$\|B_\theta^\varepsilon - c_\theta\|_{L^2(Q,d\mu)} \leq \varepsilon C \|F\|_{L^2(Q,d\mu)},$$

where the constant vector c_θ is defined as the solution of the vector equation

$$\theta \times A^{\text{hom}}(\theta \times c_\theta) + c_\theta = \int_Q F d\mu.$$

Applying back the Floquet transform, we provide the operator-norm resolvent estimate in the whole space $\mathbb{R}^3$ for the magnetic induction B_ε solution of (8) as follows:

Corollary. *(See Corollary 4.4.3.) There exists a constant $C > 0$ independent of ε and f^ε such that the following estimate holds:*

$$\|B_\varepsilon - B_\varepsilon^{\text{hom}}\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \leq \varepsilon C \|f^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)},$$

where $B_\varepsilon^{\text{hom}}$ is the solution of the homogenised equation

$$\operatorname{curl} A^{\text{hom}}(\operatorname{curl} B_\varepsilon^{\text{hom}}) + B_\varepsilon^{\text{hom}} = f^\varepsilon.$$

Note that the same result holds for the magnetic field H_ε which coincides with the magnetic induction B_ε in this case. In order to prove the theorem we construct an asymptotic approximation in power of ε for B_θ^ε solution of (13), and we provide the technical argument to obtain the uniform bound for the reminder. In Section 4.4 we discuss the estimates for the electric field E_ε and the electric displacement D_ε. In fact, using the asymptotic approximation constructed for B_θ^ε, it is possible to provide estimates for the transformed electric field and the transformed electric induction. Applying back the Floquet transform we obtain uniform estimates for D_ε and E_ε in the whole space $\mathbb{R}^3$ with respect the measure μ^ε. The main difference with the result proved for the magnetic field is that these estimates contain terms rapidly oscillating as $\varepsilon \to 0$. The approximation term here is the sum of the solution of the homogenised system (that is $E_\varepsilon^{\text{hom}}$ and $D_\varepsilon^{\text{hom}}$), and an extra element depending on $y \in Q$.

Chapter 5 is focused on obtaining operator-norm resolvent estimates for the problem of system of Maxwell equations with non-zero external current and relative magnetic permeability set to unity (that is $f^\varepsilon = 0$ and $\widetilde{A} = I$ in (6)). This result is in the first part of the work by Cherednichenko and D'Onofrio [21]. In Section 5.1 we introduce the problem (9) written in terms of the electric displacement. The analysis is focused on the following family of operators parametrised by the quasimomentum θ obtained via the Floquet transform:

$$\varepsilon^{-2} A^{1/2} \overline{e_{\varepsilon\theta}} \operatorname{curl} \operatorname{curl}(e_{\varepsilon\theta} A^{1/2} D_\theta^\varepsilon) + D_\theta^\varepsilon = -A^{1/2} G, \quad \overline{e_{\varepsilon\theta}} \operatorname{div}(e_{\varepsilon\theta} G) = 0, \tag{14}$$

where $G \in L^2(Q, d\mu)$. Section 5.2 is devoted to developing a suitable Helmholtz decomposition and a related Poincaré-type inequality. The challenge here is to take into account the quasiperiodicity of the functions involved, the arbitrary measure μ and the particular structure of the problem (14). Note that in this setting the inverse of the relative electric permittivity A plays an important role, in fact we represent the space $L^2(Q, d\mu)$ as the sum of $\operatorname{curl}(e_{\varepsilon\theta} A^{1/2})$-free and $\operatorname{div}(e_{\varepsilon\theta} A^{-1/2})$-free functions.

In Section 5.3 we formulate the main result of the chapter, the operator-norm resolvent estimate for the solution of the problem (14). In this case we uniformly bound the difference between D_θ^ε and a vector function depending on ε as follows:

Theorem. *(See Theorem 5.3.1.) The following estimate holds with a constant $C > 0$ independent of ε, θ and G:*

$$\left\| D_\theta^\varepsilon - A^{-1/2}\big(\overline{e_{\varepsilon\theta}} \nabla (e_{\varepsilon\theta} \Psi_{\varepsilon\theta}) + I\big) d_\theta^\varepsilon \right\|_{L^2(Q,d\mu)} \leq C\varepsilon \|G\|_{L^2(Q,d\mu)}.$$

The vector $d_\theta^\varepsilon \in \mathbb{C}^3$ is defined as the solution of

$$i\theta \times (i\theta \times d_\theta^\varepsilon) + \widehat{A}_{\varepsilon\theta}^{\text{hom}} d_\theta^\varepsilon = -\int_Q G,$$

and $\Psi_{\varepsilon\theta}$ is a vector function with components in $H^1_\#(Q, d\mu)$ such that

$$\overline{e_{\varepsilon\theta}} \operatorname{div} A^{-1}(\nabla (e_{\varepsilon\theta} \Psi_{\varepsilon\theta}) + e_{\varepsilon\theta} I) = 0, \quad \int_Q \Psi_{\varepsilon\theta} \, d\mu = 0.$$

Note that the homogenised matrix is defined as

$$\widehat{A}_{\varepsilon\theta}^{\text{hom}} := \int_Q A^{-1}(\overline{e_{\varepsilon\theta}} \nabla (e_{\varepsilon\theta} \Psi_{\varepsilon\theta}) + I) d\mu,$$

and depends on $\varepsilon\theta$.

Applying back the Floquet transform we obtain the following result in the whole-space $\mathbb{R}^3$ with measure μ^ε.

Corollary. *(See Corollary 5.3.6.) Let $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ and denote $g_\theta^\varepsilon := \overline{e_{\varepsilon\theta}} \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon g(x)$*

so that

$$\int_Q g_\theta^\varepsilon d\mu = \widehat{g}(\theta), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \widehat{g}(\theta) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} g\overline{e_\theta}\, d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.$$

There exists a constant $C > 0$ such that the following estimate holds for D_ε

$$\Big\|D_\varepsilon - (2\pi)^{-3/2}\int_{\mathbb{R}^3} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)\big(\widehat{A}^{\mathrm{hom}}_{\varepsilon\theta}\big)^{-1}\big(\mathfrak{A}^{\mathrm{hom}}_\theta + I\big)^{-1}\widehat{g}(\theta)e_\theta d\theta\Big\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)} \\ \leq C\varepsilon\|g^\varepsilon\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)}.$$

$\forall\varepsilon > 0$*. Here $\mathfrak{A}^{hom}_\theta$ is the matrix valued quadratic form given by the definition of d^ε_θ, and $\Psi_{\varepsilon\theta}$ is the vector function defined as above, for all values $\theta \in \mathbb{R}^3$.*

We uniformly approximate the electric displacement D_ε, solution of (9), with a pseudo-differential operator with a two-scale symbol depending on $\varepsilon\theta$ and θ.

From a formal point of view, the pseudo-differential operator can be written as an infinite order series in powers of ε. Such a series is not rigorous, in fact if we try to truncate it at some order of ε, we lose part of its meaning. The reason for it resides in the structure of the operator: for every ε we have an integral with respect to θ which has a role in the above estimate. To understand the structure of the pseudo-differential operator, we analyse the first element of the formal series. Setting $\varepsilon = 0$, we obtain a standard construction: the approximation term is the sum of $D^{\mathrm{hom}}_\varepsilon$, the solution of the formal homogenised equation, and an element rapidly oscillating when ε goes to 0. Hence, the first term of the formal series has the same structure as the limit term obtained in the case of Maxwell equations system with magnetic permeability set to unity and zero external currents. The high-order terms are solutions of some singular perturbed problems, and they are all contributive for the approximation of D_ε (see Section 5.3.2 for the full discussion).

In Section 5.4 we provide the proof of the main theorem of the chapter, based on the construction of an asymptotic approximation in powers of ε for D^ε_θ, and we obtain an estimate uniform in θ for the reminder. Furthermore we formulate an analogous estimate result for the magnetic field H_ε and magnetic induction B_ε.

In Chapter 6 we analyse the full system of Maxwell equations, where the relative magnetic permeability is an arbitrary matrix-valued function. This result is in the second part of the work [21] by Cherednichenko and D'Onofrio. In Section 6.1 we introduce the formulation of the system of Maxwell equations starting from (6), and without loss of generality we set $f^\varepsilon = 0$. The analysis is focused on the problem (10) written in terms of electric displacement. The aim is to study the following family of operators obtained via the Floquet transform:

$$\varepsilon^{-2}A^{1/2}\overline{e_{\varepsilon\theta}}\,\mathrm{curl}\,\widetilde{A}\,\mathrm{curl}(e_{\varepsilon\theta}A^{1/2}D^\varepsilon_\theta) + D^\varepsilon_\theta = -A^{1/2}G, \quad \overline{e_{\varepsilon\theta}}\,\mathrm{div}(e_{\varepsilon\theta}G) = 0, \tag{15}$$

for $G \in L^2(Q, d\mu)$. In Section 6.2 we write the asymptotic approximation in powers of ε

for D_θ^ε and we formulate the main result (see Theorem 6.2.3 and Corollary 6.2.5), which has the same structure as the one obtained in Chapter 5 for the non-zero current case. In fact, the dependence on ε in the estimates is a consequence of the geometry of the space. In the case of the general Maxwell system, the space $L^2(Q, d\mu)$ is represented as the sum of functions which are $\operatorname{curl}(e_{\varepsilon\theta}A^{1/2})$-free and $\operatorname{div}(e_{\varepsilon\theta}A^{-1/2})$-free. Hence, the presence of $\widetilde{A}$ in the equations (10) and (15) does not influence the structure of the estimates.

It is important to note that $\widetilde{A}^{\rm hom}$, the homogenised matrix linked to $\widetilde{A}$, has a different structure with respect to $\widehat{A}_{\varepsilon\theta}^{\rm hom}$. In fact $\widetilde{A}^{\rm hom}$ does not depend on $\varepsilon\theta$, it has constant coefficients and does not change the nature of the approximation term. In Section 6.3 we prove the uniform norm resolvent estimate for D_θ^ε. The analytical approach for the general Maxwell system combines the strategy created for the case of non-magnetic Maxwell system with zero external currents (that is Chapter 4) and the tools developed for the analysis of the non-magnetic Maxwell system with external currents (that is Chapter 5). Operator-norm resolvent estimates are provided as well for the magnetic field H_ε and the magnetic induction B_ε.

In this thesis every chapter starts with a brief introduction containing the aim of the research developed in the chapter. Proofs of lemmas, propositions and theorems end with the symbol $\square$. The symbols ":=" and "$\doteq$" are used to denote the expression on the right-hand side of the symbol by its left-hand side.

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Chapter 1

Elliptic equations in functions spaces with respect to Borel measures: the scalar case

Introduction

In this chapter we start the analysis of a scalar elliptic problem emerging from a second order partial differential equation with rapidly oscillating coefficients, in the setting of singular periodic structures. In the first place, we introduce the mathematical framework necessary to understand the problem in this particular geometric setting. Furthermore, we adapt two classical homogenisation techniques developed in the case of Lebesgue measure, to the case of arbitrary Borel measures.

Labelling with Q the periodicity cell, a singular periodic object is a Q-periodic set of lower dimension than the ambient space $\mathbb{R}^d$, $d \in \mathbb{N}$. Each singular object carries a natural measure μ supported on the structure itself. Classical examples are a straight line segment periodically extended to the whole plane, a square net or a system of parallel wires. In general, a singular periodic object is described by a periodic Borel measure μ with periodicity cell $Q = [0, 1)^d$, such that $\int_Q d\mu = 1$. To formulate the homogenisation problem, we introduce the parameter ε and the measure μ_ε by setting

$$\mu^\varepsilon(B) = \varepsilon^d \mu(\varepsilon^{-1} B), \tag{1.1}$$

for all Borel sets $B \subset \Omega$, where $\Omega \subseteq \mathbb{R}^d$ is an open subset. The measure μ^ε is ε-periodic and we have

$$\int_{\varepsilon Q} d\mu^\varepsilon = \varepsilon^d \int_Q d\mu = \varepsilon^d.$$

It follows that the measure μ^ε weakly converges to the Lebesgue measure, that is

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\phi\, d\mu^\varepsilon = \int_{\mathbb{R}^d}\phi\, dx, \quad \forall\phi\in C_0^\infty(\mathbb{R}^d).$$

As pointed out by Zhikov in [71] and then discussed in more details in [72] and [73], the analysis of the elliptic problem on singular periodic structures needs a careful description of the property of differentiability of functions square integrable with respect to an arbitrary Borel measure.

In Section 1.1 we introduce the so called Sobolev spaces with respect to an arbitrary Borel measure and we define their properties. In particular we analyse the weak definitions of gradient and divergence with respect to an arbitrary Borel measure.

In Section 1.2 we focus our attention on the homogenisation of a scalar elliptic equation problem in the setting of periodic singular structures. The first technique we analyse is the two-scale convergence, introduced by Nguetseng in [51] and then developed and used in applications to problems in homogenisation by Allaire [1]. This technique is self-contained: in a single process it is possible to construct the homogenised equation and to prove the convergence. The idea of the two-scale convergence is to preserve in the limit the information about oscillations of the elements of a sequence on an ε scale. In what follows we analyse the idea developed by Kamotski and Smyshlyaev in [36], where they study high contrast equations in the Lebesgue measure setting, with the two-scale convergence technique. Our aim is to adapt this method to the elliptic problem in the setting of arbitrary Borel measures.

Section 1.3 is devoted to the construction of the homogenised equation for the scalar elliptic problem in the setting of singular periodic structures through the method of multiple-scales asymptotic expansions. This technique has been originally used by Sanchez–Palencia [52] which only did formal asymptotics, and by Bakhvalov [3], [4] which provided error estimates, both in the case of Lebesgue measure. After them, similar results had been obtained by Bensoussan, Lions and Papanicolaou [6] and Keller [39]. This approach has been applied to the general theory of homogenisation in Sanchez-Palencia [53] and Bakhvalov and Panasenko [2]. For the setting of arbitrary Borel measures we only provide formal asymptotics, in fact we do not prove any convergence in this section.

1.1 The set of gradients with respect to a measure

The aim of this section is to define the mathematical framework necessary to analyse a scalar elliptic problem on singular periodic structures. In particular we carefully describe the property of differentiability of a square integrable function with respect an arbitrary Borel measure. We start with the definition of Sobolev spaces with the measure μ defined above.

Definition 1.1.1. *The space $H^1_\# = H^1_\#(Q, d\mu)$ is defined as the closure of the set of* *pairs* $\{(\phi, \nabla\phi), \phi \in C^\infty_\#(Q)\}$ *in the product* $L^2(Q, d\mu) \times L^2(Q, d\mu; \mathbb{C}^d)$*, where* $C^\infty_\#(Q) = C^\infty_\#$ *denotes the set of* Q*-periodic* $C^\infty(\mathbb{R}^d)$ *functions.*

Elements of this closure are pairs (u, v) *where* u *is a scalar function and* v *is a vector-valued function such that*

$$\exists \phi_n \in C^\infty_\#(Q) \qquad \int_Q |u - \phi_n|^2 d\mu \to 0, \qquad \int_Q |v - \nabla\phi_n|^2 d\mu \to 0. \tag{1.2}$$

As a particular case of Definition 1.1.1 we say that g is a μ-gradient of zero, and write $g \in \Gamma^\mu(0)$ whenever

$$\exists \phi_n \in C^\infty_\#(Q) \qquad \int_Q |\phi_n|^2 d\mu \to 0, \qquad \int_Q |g - \nabla\phi_n|^2 d\mu \to 0. \tag{1.3}$$

The component v in equation (1.2) is referred to as a μ-gradient of the function u. As we see below, a function $u \in L^2(Q, d\mu)$ may have more than one μ-gradient. We use the notation $\Gamma^\mu(u)$ for the set of μ-gradients of u. Whenever we deal with an arbitrary element v of the set $\Gamma^\mu(u)$ we write $v \in \Gamma^\mu(u)$, and we use the notation $\nabla^\mu u$ when we indicate a specific μ-gradient.

Note that in the case when μ is the Lebesgue measure, for all $u \in L^2(Q, d\mu)$ the set $\Gamma^\mu(u)$ consists of one element $\nabla^\mu u$, and we continue using the standard notation ∇u for this element.

Furthermore, note that the set of μ-gradients of $u \in H^1_\#(Q, d\mu)$ has the linear structure of the subspace $\Gamma^\mu(0)$ "shifted" by a gradient $\nabla^\mu u$.

Proposition 1.1.1. *For all* $u \in H^1_\#(Q, d\mu)$ *and any* $\nabla^\mu u \in \Gamma^\mu(u)$*, one has*

$$\Gamma^\mu(u) = \nabla^\mu u + \Gamma^\mu(0), \tag{1.4}$$

where the right-hand side denotes the set $\{\nabla^\mu u + w : w \in \Gamma^\mu(0)\}$*.*

Proof. The equality (1.4) is understood in the sense that for $\nabla^\mu u,\ v \in \Gamma^\mu(u)$ one has $v - \nabla^\mu u \in \Gamma^\mu(0)$. In order to verify it, notice that according to Definition 1.1.1 there exist sequences $\{\phi_n\}$, $\{\tilde{\phi}_n\} \subset C^\infty_\#(Q)$ such that (1.2) holds and

$$\int_Q |u - \tilde{\phi}_n|^2 d\mu \to 0, \qquad \int_Q |\nabla^\mu u - \nabla\tilde{\phi}_n|^2 d\mu \to 0.$$

The element $\{\phi_n - \tilde{\phi}_n\}$ provides the approximating sequence for $v - \nabla^\mu u$. Indeed, one has $\int_Q |\phi_n - \tilde{\phi}_n|^2 d\mu \to 0$ and

$$\begin{aligned}\int_Q |(v - \nabla^\mu u) - \nabla(\phi_n - \tilde{\phi}_n)|^2 d\mu &= \int_Q |(v - \nabla \phi_n) - (\nabla^\mu u - \nabla \tilde{\phi}_n)|^2 d\mu \\ &\leq \int_Q |v - \nabla \phi_n|^2 d\mu + \int_Q |\nabla^\mu u - \nabla \tilde{\phi}_n|^2 d\mu \to 0,\end{aligned}$$

as required. □

While in (1.4) one can take any μ-gradient $\nabla^\mu u$, the choice is unique by imposing a further constraint, that is $\nabla^\mu u \in \Gamma^\mu(0)^\perp$, where $\Gamma^\mu(0)^\perp$ is the orthogonal complement of $\Gamma^\mu(0)$ in the $L^2(Q, d\mu)$ sense.

Proposition 1.1.2. *For all $u \in L^2(Q, d\mu)$ there exists a unique $w \in \Gamma^\mu(u) \cap \Gamma^\mu(0)^\perp$ such that $\Gamma^\mu(u) = w + \Gamma^\mu(0)$.*

Proof. First we show that $\Gamma^\mu(0)$ is closed, hence $L^2(Q, d\mu) = \Gamma^\mu(0) \oplus \Gamma^\mu(0)^\perp$. Indeed let $\{y_n\} \subset \Gamma^\mu(0)$ such that $\int_Q |y_n - y|^2 d\mu \to 0$. If $y \in \Gamma^\mu(0)$, then $\Gamma^\mu(0)$ is closed. We know by (1.3) that $\exists \psi_n \in C^\infty_\#(Q)$ such that $\|\psi_n\|^2_{L^2(Q,d\mu)} \to 0$ and $\|\nabla \psi_n - y_n\|^2_{L^2(Q,d\mu)} \to 0$. But also $\|\nabla \psi_n - y\|^2_{L^2(Q,d\mu)} \to 0$, because

$$\begin{aligned}\|\nabla \psi_n - y\|^2_{L^2(Q,d\mu)} &= \|\nabla \psi_n - y_n + y_n - y\|^2_{L^2(Q,d\mu)} \\ &\leq \|\nabla \psi_n - y_n\|^2_{L^2(Q,d\mu)} + \|y_n - y\|^2_{L^2(Q,d\mu)} \to 0,\end{aligned}$$

so $y \in \Gamma^\mu(0)$.

For the existence take any $\tilde{w} \in \Gamma^\mu(u)$. We know that $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$ where $\tilde{w}_1 \in \Gamma^\mu(0)$ and $\tilde{w}_2 \in \Gamma^\mu(0)^\perp$. Note that $\tilde{w}_2 = \tilde{w} - \tilde{w}_1 \in \Gamma^\mu(u)$ by an argument similar to the proof of Proposition 1.1.1, hence $\tilde{w}_2 \in \Gamma^\mu(u) \cap \Gamma^\mu(0)^\perp$. Finally note that by Proposition 1.1.1 there exists $\tilde{v} \in \Gamma^\mu(0)$ such that $u = \tilde{w} + \tilde{v}$, and hence $u = \tilde{w}_2 + \tilde{v}_2 + \tilde{w}_1$, where clearly $\tilde{v}_2 + \tilde{w}_1 \in \Gamma^\mu(0)$.

The uniqueness is proved by contradiction. Indeed let $w, v \in \Gamma^\mu(u) \cap \Gamma^\mu(0)^\perp$, then $g = v - w$ is an element of $\Gamma^\mu(0)$, since $w, v \in \Gamma^\mu(u)$ and using an argument similar to the proof of Proposition 1.1.1. On the other hand, by linearity of $\Gamma^\mu(0)^\perp$ one has $g = v - w \in \Gamma^\mu(0)^\perp$, hence $g = 0$ as the only element in $\Gamma^\mu(0) \cap \Gamma^\mu(0)^\perp$. □

Finally, we note that the components in $\Gamma^\mu(0)^\perp$ of any two elements in $\Gamma(u)$ coincide: for a given $u \in H^1_\#(Q, d\mu)$, and $z_1,\ z_2 \in \Gamma^\mu(u)$, let $z_i = z_i^\perp + z_i^{\|}$, where $z_i^\perp \in \Gamma^\mu(0)^\perp$ and $z_i^{\|} \in \Gamma^\mu(0)$ for $i = 1, 2$, then $z_1^\perp = z_2^\perp$.

1.1.1 The μ-divergence, ergodicity and Poincaré inequality

In what follows, we consider elliptic equations in divergence form in Sobolev spaces with respect to an arbitrary measure. In view of the possible non-uniqueness of the μ-gradient of functions in such spaces, one can follow different procedures for making rigorous sense of the divergence operator that appears in the equations. For example, one can define the divergence as the trace of some (matrix-valued) μ-gradient of the relevant vector field. We adopt a different approach which is better suited to the task at hand from the variational perspective.

Definition 1.1.2. *We say that a vector $z \in L^2(Q, d\mu; \mathbb{C}^d)$ and a function $g \in L^2(Q, d\mu)$ are connected by the relation $\operatorname{div}_y^\mu z = g$ if and only if*

$$\int_Q (z \cdot \nabla_y \phi + g\phi) d\mu = 0 \quad \forall \phi \in C_\#^\infty(Q). \tag{1.5}$$

In order to replace integration by parts with a comparable formula which works with measure μ, we link the μ-grandients and the μ-divergence, so we define a new space $\widetilde{H}_\#^1(Q, d\mu)$.

Definition 1.1.3. *We say that the pair (u, v) where $u \in L^2(Q, d\mu)$ and $v \in L^2(Q, d\mu; \mathbb{C}^d)$ belongs to the space $\widetilde{H}_\#^1(Q, d\mu)$ if*

$$\int_Q ug d\mu = -\int_Q v \cdot z d\mu, \quad \text{where } \operatorname{div}_y^\mu z = g. \tag{1.6}$$

Remark 1.1.4. *As proved in [73, Thm. 1], the spaces $\widetilde{H}_\#^1(Q, d\mu)$ and $H_\#^1(Q, d\mu)$ coincide.*

In order to take advantage of the weak formulation (1.5) in solving elliptic problems where v represents a (linear) function of a μ-gradient of u, we use the following property of ergodicity which, in some sense, turns out to be "sufficient for homogenisation".

Definition 1.1.5. *The measure μ is ergodic if $u =$ constant μ-a.e. whenever there exists $u_n \in C_\#^\infty(Q)$ s.t.*

$$\int_Q |u_n - u|^2 d\mu \to 0, \qquad \int_Q |\nabla u_n|^2 d\mu \to 0.$$

The ergodicity means that u is constant if it belongs to $H_\#^1(Q, d\mu)$ and one of its μ-gradients is zero.

An important tool we need to introduce in the homogenisation for the elliptic equation, is a Poincaré-type inequality. Define the space

$$V = \{v \in H_\#^1(Q, d\mu) \ : \ \nabla^\mu v = 0\},$$

and denote with $V^{\perp}$ the space orthogonal in an L^2 sense to V. The Poincaré-type inequality states that there exists a constant $C > 0$ such that

$$\|P_{V^{\perp}} u\|_{L^2(Q,d\mu)} \leq C\|\nabla^{\mu} u\|_{L^2(Y,d\mu)},$$

where $P_{V^{\perp}}$ is the orthogonal L^2 projector on $V^{\perp}$.
Such inequality holds in Sobolev spaces with arbitrary measure if we assume first of all the ergodicity for the measure μ (see Definition 1.1.5). Furthermore, following the idea in the work [77, Section 5] of Zhikov and Pastukova, we assume that the embedding

$$H^1_{\#}(Q, d\mu) \subset L^2(Q, d\mu)$$

is compact. With this assumption, the spectrum of the elliptic operator $-\operatorname{div}^{\mu} \nabla^{\mu}$ on Q is discrete and the zero eigenvalue is simple.

1.2 Two-scale convergence analysis in the setting of arbitrary periodic measures

The aim of this section is to analyse the homogenised equation for the scalar problem (1.7) using the method of two-scale convergence approach presented in [36] by Kamotski and Smyshlyaev for high contrast PDE system with periodic coefficients in the setting of Lebesgue measure. Their strategy can be extended in a relative straightforward way to the setting of arbitrary Borel measure.

However, the deeper purpose of this section is to use the method of [36] to understand the structure of the leading-order term in the two-scale asymptotic and, in particular, its relation to the Helmholtz decomposition of square integrable functions.

Let $u^{\varepsilon} \in H^1(\Omega, d\mu^{\varepsilon})$ where $\Omega \subseteq \mathbb{R}^d$, such that

$$\mathcal{A}^{\varepsilon} u^{\varepsilon} \doteq -\operatorname{div} A^{\varepsilon} \nabla u^{\varepsilon} + u^{\varepsilon} = f^{\varepsilon} \in L^2(\Omega, d\mu^{\varepsilon}), \tag{1.7}$$

with Dirichelet boundary conditions. The singular measure μ^{ε} is defined in (1.1) and $A^{\varepsilon}(\cdot) = A(\cdot/\varepsilon)$ is a measurable, periodic, bounded, positive definite matrix-valued function satisfying the ellipticity condition

$$\gamma|\xi|^2 \leq A^{\varepsilon}(x)\xi \cdot \xi \leq \gamma^{-1}|\xi|^2 \quad \forall \xi \in \Omega, \quad x \in \Omega, \quad \gamma > 0.$$

Throughout the chapter we drop the superscript ε in f^{ε} for brevity.

For a fixed $\varepsilon > 0$ the boundary value problem (1.7) is understood in the sense

$$\int_{\Omega} A^{\varepsilon} \nabla u^{\varepsilon} \cdot \nabla \phi d\mu^{\varepsilon} + \int_{\Omega} u^{\varepsilon} \phi d\mu^{\varepsilon} = \int_{\Omega} f \phi d\mu^{\varepsilon} \qquad \forall \phi \in C_0^{\infty}(\Omega), \tag{1.8}$$

where C_0^∞ is the set of infinitely smooth functions with compact support.

1.2.1 A priori estimates and two-scale convergence

The approach of [36] is based on the notion of two-scale convergence in spaces with arbitrary periodic measures (see [71]):

Definition 1.2.1. *A sequence u^ε is weakly two-scale convergent to $u(x,y) \in L^2(\Omega; Q, d\mu)$ i.e. $u_\varepsilon \overset{2}{\rightharpoonup} u(x,y)$ if*

$$\int_\Omega \phi(x, \varepsilon^{-1}x) u^\varepsilon(x) d\mu^\varepsilon \to \int_\Omega \int_Q \phi(x,y) u(x,y) dx d\mu(y),$$

$\forall \phi(x,y) = a(x)b(y)$ where $a(x) \in C_0^\infty(\Omega)$ and $b(y) \in C_\#^\infty(Q)$.

In what follows we often write $d\mu$ instead of $d\mu(y)$ in the integrals, for brevity. In order to prove the two-scale convergence for the solution u^ε, we derive a priori estimates.

Lemma 1.2.2. *The following a priori estimates hold*

$$\|u^\varepsilon\|_{L^2(\Omega, d\mu^\varepsilon)} \leq C\|f\|_{L^2(\Omega, d\mu^\varepsilon)}, \tag{1.9}$$

$$\|\nabla u^\varepsilon\|_{L^2(\Omega, d\mu^\varepsilon)} \leq C\|f\|_{L^2(\Omega, d\mu^\varepsilon)}, \tag{1.10}$$

$$\|A^{\frac{1}{2}} \nabla u^\varepsilon\|_{L^2(\Omega, d\mu^\varepsilon)} \leq C\|f\|_{L^2(\Omega, d\mu^\varepsilon)}, \tag{1.11}$$

where with C we indicate a general constant that can change from line to line.

Proof. Starting from the weak formulation (1.8), we choose $\phi = u_\varepsilon$. So we have

$$\int_\Omega A^\varepsilon |\nabla u^\varepsilon|^2 d\mu^\varepsilon + \int_\Omega |u^\varepsilon|^2 d\mu^\varepsilon = \int_\Omega f u^\varepsilon d\mu^\varepsilon.$$

Note that all the terms on the left hand side are not negative. We can estimate the right hand side in the following way

$$\int_\Omega f u^\varepsilon d\mu^\varepsilon \leq \frac{1}{2}\gamma \|u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} + \frac{1}{2\gamma}\|f\|^2_{L^2(\Omega, d\mu^\varepsilon)}.$$

So we have

$$\gamma \|\nabla u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} + \|u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} \leq \frac{1}{2}\gamma \|u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} + \frac{1}{2\gamma}\|f\|^2_{L^2(\Omega, d\mu^\varepsilon)},$$

from which follow (1.9) and (1.10). To obtain (1.11), recall that $\int_\Omega A\nabla u^\varepsilon \cdot \nabla u^\varepsilon d\mu^\varepsilon = \|A^{\frac{1}{2}} \nabla u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)}$. □

For the periodicity cell we introduce the following closed linear subspace of $H^1_\#(Q, d\mu)$

$$V \doteq \{v \in H^1_\#(Q, d\mu)^d : \ \nabla^\mu v = 0\}, \tag{1.12}$$

and the following closed linear subspace of $L^2(Q, d\mu)^d$

$$W \doteq \{w \in L^2(Q, d\mu)^d : \ \mathrm{div}^\mu(A^{\frac{1}{2}} w) = 0 \text{ in } H^1_\#(Q, d\mu)\}, \tag{1.13}$$

where the $\mathrm{div}^\mu(A^{\frac{1}{2}} w)$ is understood in sense of Definition 1.1.2.

The a priori estimates, via adapting accordingly the properties of the two-scale convergence, imply the following Lemma.

Lemma 1.2.3. *There exist $u_0(x, y) \in L^2(\Omega; V)$ and $\xi_0(x, y) \in L^2(\Omega; W)$ such that, up to extracting a subsequence in ε,*

$$u^\varepsilon \stackrel{2}{\rightharpoonup} u_0(x, y), \tag{1.14}$$

$$\nabla u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla_y u_0(x, y), \tag{1.15}$$

$$A^{\frac{1}{2}} \nabla u^\varepsilon \stackrel{2}{\rightharpoonup} \xi_0(x, y). \tag{1.16}$$

Proof. The a priori estimates (1.9)-(1.11) imply, up to extracting a subsequence in ε, the existence of the two-scale limits in the measure case as proved in [71, Prop. 2.2]. So $\exists u_0(x, y), \ \xi_0(x, y) \in L^2(\Omega; Q, d\mu)$ such that satisfy (1.14)-(1.16). We show that for x a.e. they are respectively in V and W.
Starting from the weak formulation (1.8), we choose $\phi(x) = \phi^\varepsilon(x) = \varepsilon\Phi(x, \varepsilon^{-1}x)$, $\forall \Phi \in C_0^\infty(\Omega; C_\#^\infty(Q))$, so we have

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \cdot \varepsilon \nabla \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon + \int_\Omega u^\varepsilon \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon = \int_\Omega f \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon.$$

For the left hand side we can use (1.16) and obtain, using the chain rule $\varepsilon\nabla\Phi(x, \varepsilon^{-1}x) = \varepsilon\nabla_x\Phi + \nabla_y\Phi$, that

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \cdot \varepsilon \nabla \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to \int_\Omega \int_Q A^{\frac{1}{2}} \xi_0(x, y) \cdot \nabla_y \Phi(x, y) d\mu dx,$$

and

$$\int_\Omega u^\varepsilon \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to 0.$$

For the right hand side we have that for $\varepsilon \to 0$

$$\int_\Omega f \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to 0, \qquad \forall \Phi \in C_0^\infty(\Omega; C_\#^\infty(Q)).$$

So $\int_\Omega \int_Q A^{\frac{1}{2}}\xi_0(x,y)\cdot\nabla_y\Phi(x,y)d\mu dx = 0$. By Definition 1.1.2 we have

$$\int_\Omega \int_Q A^{\frac{1}{2}}\xi_0(x,y)\cdot\nabla_y\Phi(x,y)d\mu dx = -\int_\Omega \int_Q \operatorname{div}_y A^{\frac{1}{2}}\xi_0(x,y)\Phi(x,y)d\mu dx = 0.$$

By density of Φ we obtain $\operatorname{div}_y A^{\frac{1}{2}}\xi_0(x,y) = 0$ for x a.e., hence $\xi_0(x,y) \in L^2(\Omega; W)$. In a similar way we check the regularity of $u_0(x,y)$, indeed using (1.15) we have

$$\int_\Omega (A^\varepsilon)^{\frac{1}{2}}\nabla u^\varepsilon \cdot \phi(x,\varepsilon^{-1}x)d\mu^\varepsilon \to \int_\Omega \int_Q A^{\frac{1}{2}}\nabla_y u_0(x,y)\cdot\phi(x,y)d\mu dx \quad \forall \phi \in C_0^\infty(\Omega; C_\#^\infty(Q)).$$

On the other hand, the equation (1.11) ensures that $\|(A^\varepsilon)^{\frac{1}{2}}\nabla u^\varepsilon(x)\|_{L^2(\Omega,d\mu^\varepsilon)} \to 0$. This implies

$$\int_\Omega \int_Q A^{\frac{1}{2}}\nabla_y u_0(x,y)\cdot\phi(x,y)d\mu dx = 0 \qquad \forall \phi \in C_0^\infty(\Omega; C_\#^\infty(Q)).$$

By density of ϕ this gives $A^{\frac{1}{2}}\nabla_y u_0(x,y) = 0$ for x a.e., so multiplying by $A^{\frac{1}{2}}$ we have $u_0(x,y) \in L^2(\Omega; V)$. $\square$

1.2.2 Auxiliary theorems for two-scale limits

A crucial point is to understand the relation between $u_0(x,y)$ and $\xi_0(x,y)$, the limit fields and the limit fluxes.

Let $(\cdot,\cdot)$ the inner product in $H^1_\#(Q,d\mu)^d$, and indicate with $V^\perp$ the orthogonal complement to V, defined by

$$V^\perp \doteq \{w \in H^1_\#(Q,d\mu) : \ (w,v) = 0 \ \forall v \in V\}. \tag{1.17}$$

The subspaces V and $V^\perp$ are closed, hence $H^1_\#(Q,d\mu) = V \oplus V^\perp$.

In order to pass to the limit in equation (1.7) and obtain the limit problem, we prove two theorems. The first one explains when a boundary value problem admits a solution well defined in $H^1_\#(Q,d\mu)$. The second one characterises the space W and its orthogonal complement. So that happens, we assume the following Poincaré-type inequality for $v \in H^1_\#(Q,d\mu)$:

$$\|P_{V^\perp}v\|_{L^2_\#(Q,d\mu)} \leq \|\nabla v\|_{L^2(Q,d\mu)}, \tag{1.18}$$

where $P_{V^\perp}$ is the orthogonal L^2 projector on $V^\perp$ and $C > 0$ is a constant.

Theorem 1.2.4. *The problem on the periodicity cell Q for $v \in H^1_\#(Q,d\mu)$*

$$-\operatorname{div}(A\nabla v) = F \in L^2(Q,d\mu),$$

or equivalently

$$\int_Q A\nabla v \cdot \nabla w d\mu = \int_Q F w d\mu \qquad \forall w \in C^\infty_\#(Q), \tag{1.19}$$

is solvable if and only if $\langle F, w\rangle = 0\ \forall w \in V$*. When this holds, the problem is uniquely solvable in* $V^\perp$*. Furthermore if* v *is a solution and* $v_1 \in V$*, then* $v+v_1$ *is also a solution. Conversely, any two solutions can only differ for* $v_1 \in V$*.*

Proof. If v is a solution of (1.19) we choose $w \in V$ so we have that

$$\langle F, w\rangle = \int_Q A\nabla v \cdot \nabla w d\mu = 0.$$

Conversely, let $\langle F, w\rangle = 0$ and look for v such that is a solution of (1.19). This is true if $w \in V$, so we have to prove it for all $w \in V^\perp$. Choosing $V^\perp$ as Hilbert space with the inherited norm $\|\cdot\|_{V^\perp}$ (i.e. the $H^1_\#$ norm), we can apply the Lax-Milgram theorem to the bilinear form

$$B(v, w) = \int_Q A\nabla v \cdot \nabla w d\mu \qquad \forall v, w \in V^\perp.$$

With standard calculation we can obtain the continuity, indeed using that $A \in L^\infty$, we have $|B(v, w)| \le C\|v\|_{H^1_\#}\|w\|_{H^1_\#}$ $\forall v, w \in V^\perp$, for some $C > 0$. By the assumption (1.18) it is possible to prove the coercivity, $B(v, v) \ge C\|v\|^2_{H^1_\#}$ $\forall v \in V^\perp$. Hence, exists a unique solution $v \in V^\perp$ for the problem $\langle F, w\rangle = B(v, w)$ $\forall w \in V^\perp$, that is problem (1.19).

To prove that if v is a solution and $v_1 \in V$, then $v + v_1$ is a solution, we can use a classical argument by contradiction. Indeed, assuming u_1 and u_2 solutions of (1.19), set $v = u_1 - u_2$ this solves the problem with $F = 0$. Choosing $w = v$ in (1.19), we have

$$\int_Q A\nabla v \cdot \nabla v d\mu = \|A^{\frac{1}{2}}\nabla v\|^2_{L^2(Y,d\mu)} = 0,$$

then $v \in V$. □

Theorem 1.2.5. *Under the assumption* (1.18)*, let* $v \in L^2(Q, d\mu)$ *such that is orthogonal to* W *in* $L^2(Q, d\mu)$*. Then exists* $u_1 \in H^1_\#(Q, d\mu)$ *such that* $v = A^{\frac{1}{2}}\nabla u_1$*. Such* u_1 *is uniquely defined on* $V^\perp$*.*

Proof. Starting from $v = A^{\frac{1}{2}}\nabla u_1$, multiply by $A^{\frac{1}{2}}$ and apply the divergence on both sides, we have

$$-\operatorname{div}(A^{\frac{1}{2}}v) = -\operatorname{div}(A\nabla u_1). \tag{1.20}$$

Setting $F \doteq -\operatorname{div}(A^{\frac{1}{2}}v)$, in order to prove the existence of solution u_1 we need to verify that $\forall w \in V$, $\langle F, w\rangle = 0$. Using the Definition 1.1.3 we have

$$\langle F, w\rangle = \int_Q A^{\frac{1}{2}} v \cdot \nabla w d\mu = 0,$$

that is zero μ-a.e. since $w \in V$. Hence exists a unique $u_1 \in V^\perp$ which solves the problem (1.20). We now verify that u_1 satisfies the characterisation of $v \in W^\perp$. We have that

$$\|v - A^{\frac{1}{2}}\nabla u_1\|^2_{L^2(Q,d\mu)} = \langle v, v - A^{\frac{1}{2}}\nabla u_1\rangle - \langle A^{\frac{1}{2}}\nabla u_1, v - A^{\frac{1}{2}}\nabla u_1\rangle \doteq S_1 + S_2.$$

However, from (1.20) follows $v - A^{\frac{1}{2}}\nabla u_1 \in W$ then, since $v \in W^\perp$, $S_1 = 0$. On the other hand, relabelling $\phi \doteq v - A^{\frac{1}{2}}\nabla u_1$, we have

$$S_2 = \int_Q A^{\frac{1}{2}}\nabla u_1 \cdot \phi d\mu = -\int_Q u_1 \operatorname{div} A^{\frac{1}{2}}\phi d\mu,$$

that is zero since $\phi \in W$. So we have proved that $\|v - A^{\frac{1}{2}}\nabla u_1\|^2_{L^2(Q,d\mu)} = 0$, that implies the characterisation of v. $\square$

1.2.3 The limit problem

We are now ready to enunciate the following property:

Lemma 1.2.6. *Let $u_0(x,y)$ and $\xi_0(x,y)$ the two-scale limits defined above. They are such that*

$$\int_\Omega\int_Q \xi_0(x,y)\cdot\Psi(x,y)dxd\mu = -\int_\Omega\int_Q u_0(x,y)\operatorname{div}_x(A^{\frac{1}{2}}\Psi(x,y))dxd\mu, \tag{1.21}$$

$\forall\Psi(x,y) \in C^\infty(\Omega; W)$.

Proof. Let $\Psi(x,y) = g(x)\psi(y) \in C_0^\infty(\Omega; W)$, for $g \in C_0^\infty(\Omega)$ and $\psi \in W$. Starting from (1.16) we have

$$\int_\Omega (A^\varepsilon)^{\frac{1}{2}}\nabla u^\varepsilon \cdot \Psi(x, \varepsilon^{-1}x)d\mu_\varepsilon \to \int_\Omega\int_Q \xi_0(x,y)\cdot\Psi(x,y)d\mu dx.$$

Using the Definition 1.1.3 on has

$$\int_\Omega (A^\varepsilon)^{\frac{1}{2}}\nabla u^\varepsilon \cdot \Psi(x,\varepsilon^{-1}x)d\mu_\varepsilon = -\int_\Omega u_\varepsilon(x)\operatorname{div}((A^\varepsilon)^{\frac{1}{2}}\Psi(x,\varepsilon^{-1}x))d\mu_\varepsilon,$$

and by (1.14)

$$\int_\Omega u^\varepsilon \operatorname{div}((A^\varepsilon)^{\frac{1}{2}}\Psi(x,\varepsilon^{-1}x))d\mu_\varepsilon \to \int_\Omega\int_Q u_0(x,y)\operatorname{div}_x(A^{\frac{1}{2}}\Psi(x,y))d\mu dx.$$

Comparing the equations we have the identity (1.21). □

Motivated by (1.21), we introduce the following linear subspace of $L^2(\Omega;V)$

$$U \doteq \{u(x,y)\in L^2(\Omega;V) : \ \exists \xi(x,y)\in L^2(\Omega;W) \text{ such that } \forall\Psi(x,y)\in C^\infty(\Omega;W)$$
$$\int_\Omega\int_Q \xi(x,y)\cdot\Psi(x,y)dxd\mu = -\int_\Omega\int_Q u(x,y)\operatorname{div}_x(A(y)^{\frac{1}{2}}\Psi(x,y))dxd\mu\}. \quad (1.22)$$

Hence we can find for any $u_0(x,y)$ the associated $\xi_0(x,y)$, so it is natural to define the operator $T : U \to L^2(\Omega;W)$ such that $Tu_0(x,y)=\xi_0(x,y)$. The explicit form for T follows from (1.21):

$$\xi_0(x,y) = Tu_0(x,y) \doteq P_W[A^{\frac{1}{2}}\nabla_x u_0(x,y)] \in L^2(\Omega;W), \quad (1.23)$$

where P_W is the L^2 orthogonal projection on W.

Note that $C_0^\infty(\Omega;V)\subset U$, thus for $u_0(x,y)\in C_0^\infty(\Omega;V)$ we have that $Tu_0(x,y)\in C_0^\infty(\Omega;W)$. Furthermore, the following corrector property holds:

Proposition 1.2.7. *Let $\phi_0\in C_0^\infty(\Omega;V)$, then exists a unique corrector $\phi_1\in C_0^\infty(\Omega;V^\perp)$ such that*

$$T\phi_0(x,y) \doteq P_W[A^{\frac{1}{2}}\nabla_x\phi_0(x,y)] = A^{\frac{1}{2}}[\nabla_x\phi_0(x,y)+\nabla_y\phi_1(x,y)], \quad (1.24)$$

and ϕ_1 is the unique solution of $\operatorname{div}_y(A(\nabla_x\phi_0+\nabla_y\phi_1))=0$.

Proof. Let $\phi_0\in C_0^\infty(\Omega;V)$, and define

$$\eta(x,y) \doteq T\phi_0(x,y) - A^{\frac{1}{2}}\nabla_x\phi_0(x,y) \in C_0^\infty(\Omega;L^2(Y)).$$

But $\eta(x,y)\in C_0^\infty(\Omega;W^\perp)$, in fact using (1.22), (1.23), and Definition 1.1.3, we have

$$\int_\Omega\int_Q \eta(x,y)\cdot\Psi(x,y)d\mu dx = 0 \qquad \forall\Psi\in C_0^\infty(\Omega;W).$$

So $\eta(x,\cdot)\in W^\perp$, and we are in the hypothesis of Theorem 1.2.5, then exists $\phi_1(x,y)\in C_0^\infty(\Omega;V^\perp)$ such that $\eta(x,y)=A^{\frac{1}{2}}\nabla_y\phi_1(x,y)$. Comparing this formula with the definition of η, we obtain the claim. □

Now we formulate the two-scale convergence result. In the equation (1.8) we choose

$\phi(x) = \phi_\varepsilon(x) = \phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x)$, so we have

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \cdot \nabla(\phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x))d\mu^\varepsilon + \int_\Omega u^\varepsilon(\phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x))d\mu^\varepsilon$$
$$= \int_\Omega f(\phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x))d\mu^\varepsilon.$$

For the right hand side we have the following two scale limit

$$\int_\Omega f(\phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x))d\mu^\varepsilon \to \int_\Omega \int_Q f\phi_0(x, y)d\mu dx.$$

For the left hand side

$$\int_\Omega u^\varepsilon(\phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x))d\mu^\varepsilon \to \int_\Omega \int_Q u_0(x, y)\phi_0(x, y)d\mu dx,$$

and for the first integral we have

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \cdot \nabla(\phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x))d\mu^\varepsilon =$$
$$\int_\Omega (A^\varepsilon)^{\frac{1}{2}} \nabla u^\varepsilon \cdot (A^\varepsilon)^{\frac{1}{2}} (\nabla_x\phi_0 + \varepsilon^{-1}\nabla_y\phi_0 + \varepsilon\nabla_x\phi_1 + \nabla_y\phi_1)d\mu^\varepsilon.$$

$(A^\varepsilon)^{\frac{1}{2}}\nabla_y\phi_0 = 0$ since $\phi_0 \in C_0^\infty(\Omega; V)$, thus passing to the limit we have by (1.16),

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \cdot \nabla(\phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi_1(x, \varepsilon^{-1}x))d\mu^\varepsilon \to \int_\Omega \int_Q \xi_0(x, y) \cdot (A^\varepsilon)^{\frac{1}{2}}(\nabla_x\phi_0 + \nabla_y\phi_1)d\mu dx.$$

Using (1.24) and (1.23), we obtain

$$\int_\Omega \int_Q \xi_0(x, y) \cdot (A^\varepsilon)^{\frac{1}{2}}(\nabla_x\phi_0 + \nabla_y\phi_1)d\mu dx = \int_\Omega \int_Q Tu_0(x, y) \cdot T\phi_0(x, y)d\mu dx.$$

So we have that (1.8) two-scale converges to

$$\int_\Omega \int_Q P_W(A^{\frac{1}{2}}\nabla_x u_0(x, y)) \cdot P_W(A^{\frac{1}{2}}(\nabla_x\phi_0(x, y))d\mu dx + \int_\Omega \int_Q u_0(x, y)\phi_0(x, y)d\mu dx$$
$$= \int_\Omega \int_Q f\phi_0(x, y)d\mu dx \qquad \forall \phi_0 \in C_0^\infty(\Omega; V). \tag{1.25}$$

This is the weak formulation of the limit problem for $u_0(x, y) \in U$.

The result obtained with this method is the two-scale convergence of u^ε, solution of the problem (1.7), to the function $u_0 \in U$, solution of the limit problem (1.25). Hence, with a single process we found the limit equation and we proved the two-scale convergence. In the next section we apply the technique of two-scale asymptotic expansion to the solution of problem (1.7), in order to construct explicitly an approximation for u^ε. The leading-order term of such approximation will have the same structure of u_0. The main difference with the present section is that, in the method of asymptotic expansion, the construction of the limit equation and the convergence of the solution of the original problem to the solution of the limit problem, are two different tasks to prove.

1.3 Asymptotic Expansion

In this section we apply the method of two-scale asymptotic expansion to the scalar elliptic problem (1.7) in the setting of singular periodic structures. This approach has been extensively used over the years in the classical setting of Lebesgue measure (see for example [4], [6], [53]). It is based on the assumption that the solution can be written as an asymptotic expansion in powers of ε, which allows to construct the homogenised equation.

The method of two-scale asymptotic expansion consists of two parts: the formal construction of the homogenised equation and the convergence of the solution of the original problem to the solution of the homogenised equation. The result of this section is formal asymptotics, in fact we are not pursuing any convergence here.

The purpose of this section is to be a bridge between the classical approaches, and our new technique developed in Chapter 2, in which we revise the classical notion of multiscale asymptotic.

1.3.1 The formal expansion

Let us consider the problem (1.7) defined in Section 1.2. In order to find an approximation for u^ε that takes into account the rapid oscillation of the coefficients of the equation, we postulate the following ansatz for u_ε:

$$u^\varepsilon(x) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n\big(x, \frac{x}{\varepsilon}\big), \tag{1.26}$$

where $u_n(x,y)$, $n = 0, 1, 2, \ldots$, is periodic in y. The "slow" variables $x \in \Omega$ measures the variations within the region of interest, the "fast" variables $y \in Q$ describes the variations within the periodic cell.

We plug $u^\varepsilon(x)$ in the equation (1.7), and using the chain rules $\operatorname{div} = \operatorname{div}_x + \varepsilon^{-1} \operatorname{div}_y$

and $\nabla = \nabla_x + \varepsilon^{-1}\nabla_y$, we obtain the following formal equation:

$$
\begin{aligned}
&\mathcal{A}^\varepsilon u^\varepsilon(x) - f(x) \\
&\quad = \varepsilon^{-2}\left(\operatorname{div}_y A\nabla_y u_0(x,y)\right) \\
&\quad + \varepsilon^{-1}\left(\operatorname{div}_y A\nabla_y u_1(x,y) + \operatorname{div}_y A\nabla_x u_0(x,y) + \operatorname{div}_x A\nabla_y u_0(x,y)\right) \\
&\quad + \varepsilon^{0}(\operatorname{div}_y A\nabla_y u_2(x,y) + \operatorname{div}_y A\nabla_x u_1(x,y) \\
&\quad + \operatorname{div}_x A\nabla_y u_1(x,y) + \operatorname{div}_x A\nabla_x u_0(x,y) + u_0(x,y) - f(x)) + o(\varepsilon) = 0.
\end{aligned} \tag{1.27}
$$

This equation holds if and only if the terms of every order of smallness of ε are zero.

1.3.2 Construction of the homogenised equation

In order to characterise the terms of the expansion and to construct the homogenised equation, we start with the analysis of the equation of power ε^{-2}, so we have

$$\operatorname{div}_y A\nabla_y u_0(x,y) = 0,$$

which is understood as

$$\int_Q A\nabla_y u_0(x,y)\cdot\nabla_y\phi(x,y)d\mu(y) = 0 \qquad \forall\phi\in C^\infty_\#(Q). \tag{1.28}$$

We choose $\phi(x,y) = u_0(x,y)$, thus using the ellipticity condition on A we have

$$\int_Q A|\nabla_y u_0(x,y)|^2 d\mu(y) = 0 \Rightarrow \nabla_y u_0(x,y) = 0.$$

Since μ is ergodic, we have $u_0(x,y) = u_0(x)$, that is u_0 is constant in y. This implies in the second equation of 1.27 that $\nabla_y u_0(x,y) = 0$, and so the ε^{-1} term satisfies

$$\operatorname{div}_y A\nabla_y u_1(x,y) = -\operatorname{div}_y A\nabla_x u_0(x),$$

which is understood as

$$\int_Q A\nabla_y u_1(x,y)\cdot\nabla_y\phi(x,y)d\mu(y) = -\int_Q A\nabla_x u_0(x)\cdot\nabla_y\phi(x,y)d\mu(y) \qquad \forall\phi\in C^\infty_\#(Q). \tag{1.29}$$

From the Theorem 1.2.4 we know that the equation (1.29) has a well defined solution if and only if

$$\langle -\operatorname{div}_y A\nabla_x u_0(x), \psi(x,y)\rangle = 0 \qquad \forall\psi\in V,$$

where V defined in (1.12). This follows by Definition 1.1.3, in fact

$$-\int_Q \operatorname{div}_y A\nabla_x u_0(x)\psi(x,y)d\mu(y) = -\int_Q A\nabla_x u_0(x)\cdot\nabla_y\psi(x,y)d\mu(y) = 0 \quad \forall\psi\in V.$$

From equation (1.29), using the separation of variable we can write the solution $u_1(x,y)$ in the following way:

$$u_1(x,y) = \sum_{k=1}^{d} N_k(y)\partial_{x_k} u_0(x) + \tilde{u}_1(x,y), \tag{1.30}$$

where $\tilde{u}_1 \in V$. To find the equation solved by the vector $N = (N_1 \ldots, N_d)$, we plug (1.30) into (1.29). We obtain that $N_k \in H^1_{\#}(Q, d\mu)$, $k = 1, \ldots, d$ solve the following cell problems:

$$\int_Q A\nabla_y N_k(y) \cdot \nabla_y \phi(x,y) d\mu(y) = -\int_Q A_{kj} \partial_{y_j} \phi(x,y) d\mu(y), \quad \forall \phi \in C^\infty_{\#}(Q), \tag{1.31}$$

where the summation with respect to j is simplified. In order to prove the existence of a solution N_k we need the coercivity of the form defined by the left hand side of (1.31). Hence we assume the Poincaré-type inequality

$$\|P_{V^\perp} N_k\|_{H^1_{\#}(Q,d\mu)} \leq C \|\nabla N_k\|_{L^2(Q,d\mu)}, \tag{1.32}$$

where $C > 0$ is a constant.

Now we study the solvability conditions of the ε^0 term in the equation (1.27).

$$\langle \operatorname{div}_y A\nabla_x u_1(x,y) + \operatorname{div}_x A\nabla_y u_1(x,y) + \operatorname{div}_x A\nabla_x u_0(x,y) + u_0(x,y) - f, \psi \rangle = 0, \ \forall \psi \in V. \tag{1.33}$$

Using the definition (1.30) for $u_1(x,y)$, one has

$$\begin{aligned} &\langle \operatorname{div}_y A\nabla_x N_k(y)\partial_{x_k} u_0(x) + \operatorname{div}_x A\nabla_y N_k(y)\partial_{x_k} u_0(x) + \operatorname{div}_x A\nabla_x u_0(x,y) \\ &+ u_0(x,y) - f(x), \psi \rangle + \langle \operatorname{div}_y A\nabla_x \tilde{u}_1 + \operatorname{div}_x A\nabla_y \tilde{u}_1, \psi \rangle = 0 \quad \forall \psi \in V. \end{aligned}$$

Furthermore, we have

$$\langle \operatorname{div}_y A\nabla_x \tilde{u}_1 + \operatorname{div}_x A\nabla_y \tilde{u}_1, \psi \rangle = 0 \quad \forall \psi \in V,$$

indeed $\langle \operatorname{div}_x A\nabla_y \tilde{u}_1, \psi \rangle = 0$ because $\tilde{u}_1 \in V$. Also $\langle \operatorname{div}_y A\nabla_x \tilde{u}_1, \psi \rangle = 0$ since by Definition 1.1.3 this is equivalent to $\int_Q A\nabla_x \tilde{u}_1 \nabla_y \psi d\mu = 0$, that is null since $\psi \in V$. Furthermore $\langle \operatorname{div}_y A\nabla_x N_k(y)\partial_{x_k} u_0(x), \psi \rangle = 0$ by Definition 1.1.3, so we obtain that

$$\langle \operatorname{div}_x A\nabla_y N_k(y)\partial_{x_k} u_0(x) + \operatorname{div}_x A\nabla_x u_0(x,y) + u_0(x,y) - f(x), \psi \rangle = 0 \quad \forall \psi \in V.$$

Thus we have that

$$\int_Q \operatorname{div}_x A\nabla_y N_k(y)\partial_{x_k} u_0(x)\psi(x,y)d\mu(y)$$
$$+\int_Q \operatorname{div}_x A\nabla_x u_0(x)\psi(x,y)d\mu(y)+\int_Q u_0(x)\psi(x,y)d\mu(y)-\int_Q f(x)\psi d\mu(y)=0 \quad \forall\psi\in V.$$

Hence, we obtain the homogenised equation

$$\operatorname{div}_x A^{\text{hom}}\nabla_x u_0(x)+u_0(x)=f(x), \tag{1.34}$$

where $A^{\text{hom}}=\int_Q A(\nabla_y N(y)+I)d\mu(y)$ is the homogenised matrix. This definition fits with the general notion of the homogenised matrix presented in the Introduction. Every time, in the thesis, this definition is consistent with what we refer to the homogenised matrix.

The problem we have carried out in this section is the formal construction of the homogenised equation for the scalar elliptic problem (1.7) in the setting of singular periodic structures. To make this process rigorous it is necessary to establish any convergence between u^ε and u_0. The formal approach presented here, is a particular case of a standard tool used for problems in homogenisation. A question arises whether the two-scale expansion leads to reasonable convergence estimates. We do not tackle the issue of convergence here, and we move on to the toolbox which brings us to a stronger result.

Rather, this section is meant to be an intermediate step between the classical methods on homogenisation and our new technique presented in the next chapter. There, we develop a method to obtain operator-norm estimates for the problem (1.7), namely we prove that there exists a constant $C>0$ independent of ε and f such that the following estimate hold:

$$\|u^\varepsilon-u_0\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}\leq \varepsilon C\|f\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}. \tag{1.35}$$

The two-scale approach that we have obtained in a formal way, motivated us to think about new ways to representing the two-scale structure of the solution. In fact, the method of Chapter 2 is based on the construction of an asymptotic approximation of the solution in power of ε. This approximation has a structure which is linked with the expansion developed in the present section, however it also crucially differs from it, in that the macroscopic variable is in some sense replaced by an additional parameter, the so-called "quasimomentum".

The estimate (1.35) tempts us to continue using the approach of Chapter 2 to provide a justification for the asymptotic expansion obtained by the classical formal construction. However, looking further ahead, it will happen to be not the case when we move to others problems.

In Chapters 3-6 we see that our method is bringing new results for the vectorial problem for the system of Maxwell equations. In particular, in Sections 3.3 and 3.4, we formally analyse the Maxwell system with the asymptotic expansion technique, and we obtain what is suggested to be the homogenised equation. But this result is not operator-norm close to the solution of the original problem, hence an estimate of the form (1.35) can be obtained only with a u_0 which is different from the standard one arising from two-scale asymptotic approach.

Chapter 2

Operator-norm convergence estimates for scalar elliptic homogenisation problems

Introduction

The goal of the present chapter is to prove order-sharp norm-resolvent convergence estimates for the elliptic operator with periodic rapidly oscillating coefficients for a wide class of underlying periodic measures. This result can be found in the work [22] by Cherednichenko and D'Onofrio. Norm-resolvent convergence in homogenisation for the "classical" problem concerning the case of Lebesgue measure goes back to the works [55], [70], where the asymptotic analysis of the Green functions of the corresponding problems is carried out, which were followed by the operator-theoretic approach of [8]. An alternative approach, based on the uniform power-series asymptotic analysis of the fibre operators in the associated direct integral, was recently developed in [26]. In the present work we adopt the overall strategy of the latter work, in the setting of an arbitrary periodic Borel measure. As pointed in Chapter 1, it is important to provide a description of Sobolev spaces with respect to arbitrary Borel measures. In what follows we briefly introduce the tools we employ, namely the Sobolev spaces of quasiperiodic functions with respect to an arbitrary Borel measure (Section 2.1) and the Floquet transform (Section 2.1.1). In Section 2.2 we formulate and prove our main result (Theorem 2.2.3). All functions spaces that we use are defined over the field $\mathbb{C}$ of complex numbers.

Consider a Q-periodic, $Q := [0,1)^d$, Borel measure μ, in $\mathbb{R}^d$ such that $\mu(Q) = 1$, and for each $\varepsilon > 0$ define an ε-periodic measure μ^ε by the formula $\mu^\varepsilon(B) = \varepsilon^d \mu(\varepsilon^{-1}B)$ for all Borel sets $B \subset \mathbb{R}^d$, $d \in \mathbb{N}$. In the present work we study the asymptotic behaviour, as $\varepsilon \to 0$, of the solutions u^ε to the problems

$$-\nabla \cdot A(\cdot/\varepsilon)\nabla u^\varepsilon + u^\varepsilon = f^\varepsilon, \qquad f^\varepsilon \in L^2(\mathbb{R}^d, d\mu^\varepsilon), \qquad \varepsilon > 0, \tag{2.1}$$

where A is a positive bounded Q-periodic μ-measurable real-valued matrix function. We aim to prove operators-norm estimates between u^ε and the solution to the homogenised equation

$$-\nabla \cdot A^{\mathrm{hom}}\nabla u^\varepsilon_{\mathrm{hom}} + u^\varepsilon_{\mathrm{hom}} = f^\varepsilon, \qquad f^\varepsilon \in L^2(\mathbb{R}^d, d\mu^\varepsilon), \tag{2.2}$$

with a constant matrix A^{hom}, *i.e.* uniform estimates of the form

$$\|u^\varepsilon - u^\varepsilon_{\mathrm{hom}}\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} \le C\varepsilon \|f^\varepsilon\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)},$$

where $C > 0$ is independent of f^ε, ε.

Solutions to (2.1) are understood as a pair $(u^\varepsilon, \nabla u^\varepsilon)$ in the space $H^1(\mathbb{R}^d, d\mu^\varepsilon)$, defined (*cf.* [77]) as the closure of the set $\{(\psi, \nabla\psi), \psi \in C_0^\infty(\mathbb{R}^d)\}$ in the norm of $L^2(\mathbb{R}^d, d\mu^\varepsilon) \oplus \left[L^2(\mathbb{R}^d, d\mu^\varepsilon)\right]^d$. For $f^\varepsilon \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$, we say that $(u^\varepsilon, \nabla u^\varepsilon) \in H^1(\mathbb{R}^d, d\mu^\varepsilon)$ is a solution to (2.1) if

$$\int_{\mathbb{R}^d} A(\cdot/\varepsilon)\nabla u^\varepsilon \cdot \overline{\nabla\psi}\, d\mu^\varepsilon + \int_{\mathbb{R}^d} u^\varepsilon \overline{\psi}\, d\mu^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon \overline{\psi}\, d\mu^\varepsilon \qquad \forall (\psi, \nabla\psi) \in H^1(\mathbb{R}^d, d\mu^\varepsilon). \tag{2.3}$$

Note that for each $\varepsilon > 0$ the left-hand side of (2.3) is an equivalent inner product on $H^1(\mathbb{R}^d, d\mu^\varepsilon)$, and its right-hand side is a linear bounded functional on $H^1(\mathbb{R}^d, d\mu^\varepsilon)$. Invoking the Riesz representation theorem (see *e.g.* [7, p. 32]) yields the existence and uniqueness of solution to (2.1).

In what follows we study the resolvent of the operator $\mathcal{A}^\varepsilon$ with domain

$$\begin{aligned}\mathrm{dom}(\mathcal{A}^\varepsilon) = \Big\{ & u \in L^2(\mathbb{R}^d, d\mu^\varepsilon) : \ \exists\, \nabla u \in \left[L^2(\mathbb{R}^d, d\mu^\varepsilon)\right]^d \text{ such that} \\ & \int_{\mathbb{R}^d} A(\cdot/\varepsilon)\nabla u \cdot \overline{\nabla\psi}\, d\mu^\varepsilon + \int_{\mathbb{R}^d} u\overline{\psi}\, d\mu^\varepsilon = \int_{\mathbb{R}^d} f\overline{\psi}\, d\mu^\varepsilon \ \ \forall (\psi, \nabla\psi) \in H^1(\mathbb{R}^d, d\mu^\varepsilon) \\ & \text{for some } f \in L^2(\mathbb{R}^d, d\mu^\varepsilon) \Big\}.\end{aligned} \tag{2.4}$$

defined by the formula $\mathcal{A}^\varepsilon u = f - u$ whenever $f \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$ and $u \in \mathrm{dom}(\mathcal{A}^\varepsilon)$ are related as in (2.4). Note that while in general for a given $u \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$ there may be more than one element $(u, \nabla u) \in H^1(\mathbb{R}^d, d\mu^\varepsilon)$, the uniqueness of solution to (2.1) implies that for each function $u \in \mathrm{dom}(\mathcal{A}^\varepsilon)$ there is exactly one gradient ∇u such that the identity in (2.4) holds.

Clearly, the operator $\mathcal{A}^\varepsilon$ is symmetric. By an argument similar to [71, Section 7.1], we infer that $\mathrm{dom}(\mathcal{A}^\varepsilon)$ is dense in $L^2(\mathbb{R}^d, d\mu^\varepsilon)$. Indeed, it follows from (2.4) that if

$f \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$, and $u, v \in \mathrm{dom}(\mathcal{A}^\varepsilon)$ are such that $\mathcal{A}^\varepsilon u + u = f$, $\mathcal{A}^\varepsilon v + v = u$, then

$$\int_{\mathbb{R}^d} f\overline{v} = \int_{\mathbb{R}^d} |u|^2. \tag{2.5}$$

The identity (2.5) implies that if f is orthogonal to $\mathrm{dom}(\mathcal{A}^\varepsilon)$ then $u = 0$, and hence $f = 0$. Furthermore, $\mathcal{A}^\varepsilon$ is self-adjoint. Indeed, suppose that $w \in \mathrm{dom}\big((\mathcal{A}^\varepsilon)^*\big) \subset L^2(\mathbb{R}^d, d\mu^\varepsilon)$, so for some $g \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$ one has

$$\int_{\mathbb{R}^d} (\mathcal{A}^\varepsilon u)\overline{w}\, d\mu^\varepsilon = \int_{\mathbb{R}^d} u\overline{g}\, d\mu^\varepsilon \qquad \forall u \in \mathrm{dom}(\mathcal{A}^\varepsilon).$$

Consider the solution v to the problem

$$\mathcal{A}^\varepsilon v + v = g + w.$$

Then for all $u \in \mathrm{dom}(\mathcal{A}^\varepsilon)$ one has

$$\int_{\mathbb{R}^d} (\mathcal{A}^\varepsilon u + u)\overline{w} = \int_{\mathbb{R}^d} u\overline{(g+w)} = \int_{\mathbb{R}^d} u\overline{(\mathcal{A}^\varepsilon v + v)} = \int_{\mathbb{R}^d} (\mathcal{A}^\varepsilon u + u)\overline{v},$$

where we use the fact that $\mathcal{A}^\varepsilon$ is symmetric and $u, v \in \mathrm{dom}(\mathcal{A}^\varepsilon)$. Since $\mathcal{A}^\varepsilon u + u$ is an arbitrary element of $L^2(\mathbb{R}^d, d\mu^\varepsilon)$, it follows that $w = v$, and in particular $w \in \mathrm{dom}(\mathcal{A}^\varepsilon)$.

Similarly, we define the operator $\mathcal{A}^{\mathrm{hom}}$ associated with the problem (2.2), so that (2.2) holds if and only if $u^0 = (\mathcal{A}^{\mathrm{hom}} + I)^{-1} f$.

All gradients, integrals and differential operators below, unless indicated explicitly otherwise, are understood appropriately with respect to the measure μ. Whenever we write $\int_Q$, we imply integration with respect to the measure μ and interchangeably use the notation and $L^2(Q, d\mu)$ and $L^2(Q)$ for the Lebesgue space of functions that are square integrable on Q with respect to μ. Throughout the paper we use the notation e_κ for the exponent $\exp(\mathrm{i}\kappa \cdot y)$, $y \in Q$, $\kappa \in [-\pi, \pi)^d$, and a similar notation e_θ for the exponent $\exp(\mathrm{i}\theta \cdot x)$, $x \in \mathbb{R}^d$, $\theta \in \varepsilon^{-1}[-\pi, \pi)^d$. We denote by $C^\infty_\#$ the set of Q-periodic functions in $C^\infty(\mathbb{R}^d)$, and $\partial_j\varphi$, $\nabla\varphi$, $\nabla(e_\kappa\varphi)$ $\nabla(e_{\varepsilon\theta}\varphi)$ stand for the classical derivatives and gradients of smooth functions ϕ, $e_\kappa\varphi$, $e_{\varepsilon\theta}\varphi$.

2.1 Sobolev spaces of quasiperiodic functions

The material of this section applies to an arbitrary Borel measure μ on Q. The following definition is motivated by [71, Section 3.1], [73].

Definition 2.1.1. *For each $\kappa \in [-\pi, \pi)^d := Q'$ we define the space H^1_κ as the closure, with respect to the natural norm of the direct sum $L^2(Q) \oplus \big[L^2(Q)\big]^d$, of the set $\big\{\big(e_\kappa\varphi, \nabla(e_\kappa\varphi)\big) : \varphi \in C^\infty_\#\big\}$. We use the notation $H^1_\#(Q, d\mu) = H^1_\#$ for the space H^1_κ,*

$\kappa = 0$. *For* $(u, v) \in H^1_\kappa$ *we keep the usual notation* ∇u *for the second element* v *in the pair.*

As discussed in [71], [73], [77], there may be different elements in H^1_κ whose first components coincide. Indeed, for any $(u, v) \in H^1_\kappa$ and a vector function w obtained as the limit in $\left[L^2(Q)\right]^d$ of the classical gradients $\nabla(e_\kappa \phi_n)$ for a sequence $\phi_n \in C^\infty_\#$ converging to zero in $L^2(Q)$ ("gradient of zero"), the pair $(u, v+w)$ is also an element of H^1_κ. Furthermore, there is a natural one-to-one mapping between H^1_κ and $H^1_\#$: for any element $(u, v) \in H^1_\kappa$ the pair $\bigl(\overline{e_\kappa}u, \overline{e_\kappa}(v - \mathrm{i}u\kappa)\bigr)$ is an element of $H^1_\#$ and for all $(\widetilde{u}, \widetilde{v}) \in H^1_\#$ one has $\widetilde{v} = \overline{e_\kappa}(v - \mathrm{i}u\kappa)$ for some $(u, v) \in H^1_\kappa$. In view of this, for $(\widetilde{u}, \widetilde{v}) \in H^1_\#$ we often write $\widetilde{v} = \nabla\widetilde{u} = \overline{e_\kappa}\nabla(e_\kappa\widetilde{u}) - \mathrm{i}\widetilde{u}\kappa$, where either $\nabla\widetilde{u}$ or $\nabla(e_\kappa\widetilde{u})$ is defined up to a gradient of zero.

Suppose that $A \in \left[L^\infty(Q, d\mu)\right]^{d\times d}$ is a pointwise positive and symmetric real-valued matrix function such that $A^{-1} \in \left[L^\infty(Q, d\mu)\right]^{d\times d}$, and for each $\kappa \in Q'$ consider the operator $\mathcal{A}_\kappa$ with domain (*cf.* (2.4))

$$\mathrm{dom}(\mathcal{A}_\kappa) = \biggl\{u \in L^2(Q) :\ \exists\, \nabla(e_\kappa u) \in \left[L^2(Q)\right]^d \text{ such that}$$
$$\int_Q A\nabla(e_\kappa u)\cdot\overline{\nabla(e_\kappa\varphi)} + \int_Q u\overline{\phi} = \int_Q F\overline{\phi} \quad \forall\phi \in C^\infty_\# \quad \text{for some } F \in L^2(Q)\biggr\}, \tag{2.6}$$

defined by the formula $\mathcal{A}_\kappa u = F - u$ whenever $F \in L^2(Q)$ and $u \in \mathrm{dom}(\mathcal{A}_\kappa)$ are related as described in the definition of $\mathrm{dom}(\mathcal{A}_\kappa)$. Notice that by the definition of H^1_κ, the set $C^\infty_\#$ of test functions in the identity in (2.6) can be equivalently replaced by H^1_κ. As discussed in the previous section for the case of operator $\mathcal{A}^\varepsilon$, since for $F = 0$ one has $u = 0$, $\nabla(e_\kappa u) = 0$, there is exactly one gradient $\nabla(e_\kappa u)$ for which (2.6) holds. Also, by an argument similar to the case of $\mathcal{A}^\varepsilon$, the domain $\mathrm{dom}(\mathcal{A}_\kappa)$ is dense in $L^2(Q)$ and $\mathcal{A}_\kappa$ is self-adjoint.

In what follows, we identify with $H^1_\#$ the set of the first components of its elements, bearing in mind that the gradient of a function in $H^1_\#$ may not be unique. We also denote by $H^1_{\#,0}$ the (closed) subspace of $H^1_\#$ consisting of functions with zero μ-mean over Q.

2.1.1 Floquet transform

In this section we define a representation for functions in $L^2(\mathbb{R}^d, d\mu)$ unitarily equivalent to the "Gelfand transform", introduced in [31] for the case of the Lebesgue measure. The properties of the Gelfand transform with respect to the measure μ are discussed in detail in [77], and here we give the definition of its Floquet version as well as the key property concerning the equation (2.1). We first define a "scaled" version of the Floquet transform (*cf.* [26]).

Definition 2.1.2. *For $\varepsilon > 0$ and $u \in C_0^\infty(\mathbb{R}^d)$, the ε-Floquet transform $\mathcal{F}_\varepsilon u$ is the function*

$$(\mathcal{F}_\varepsilon u)(z,\theta) = \left(\frac{\varepsilon}{2\pi}\right)^{d/2} \sum_{n\in Z^d} u(z+\varepsilon n)\exp(-\mathrm{i}\varepsilon n\cdot\theta), \quad z\in\varepsilon Q,\ \theta\in\varepsilon^{-1}Q'.$$

The mapping $\mathcal{F}_\varepsilon$ preserves the norm, in the sense that

$$\|\mathcal{F}_\varepsilon u\|_{L^2(\varepsilon^{-1}Q'\times\varepsilon Q, d\theta\times d\mu^\varepsilon)} = \|u\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)},$$

and it can therefore be extended to an isometry $\mathcal{F}_\varepsilon : L^2(\mathbb{R}^d, d\mu^\varepsilon) \mapsto L^2(\varepsilon^{-1}Q'\times\varepsilon Q, d\theta\times d\mu^\varepsilon)$, for which we use the same term ε-Floquet transform. Note that the inverse of $\mathcal{F}_\varepsilon$ is given by

$$(\mathcal{F}_\varepsilon^{-1}g)(z) = \left(\frac{\varepsilon}{2\pi}\right)^{-d/2} \int_{\varepsilon^{-1}Q'} g(\theta,z)\,d\theta, \quad z\in\mathbb{R}^d, \qquad g\in L^2(\varepsilon^{-1}Q'\times\varepsilon Q, d\theta\times d\mu^\varepsilon), \tag{2.7}$$

where for each $\theta\in\varepsilon^{-1}Q'$ the function $g\in L^2(\varepsilon^{-1}Q'\times\varepsilon Q, d\theta\times d\mu^\varepsilon)$ is extended as θ-quasiperiodic function to the whole of $\mathbb{R}^d$ so that

$$g(\theta,z) = \widetilde{g}(\theta,z)\exp(\mathrm{i}z\cdot\theta), \quad z\in\mathbb{R}^d, \qquad \widetilde{g}(\theta,\cdot) \quad \varepsilon Q\text{-periodic}.$$

Indeed, for all such functions g the right-hand side (2.7) is well defined and returns a function in $L^2(\mathbb{R}^d)$, *cf.* [77]:

$$\|\mathcal{F}_\varepsilon^{-1}g\|^2_{L^2(\mathbb{R}^d)} = \sum_{n\in\mathbb{Z}^d}\|(\mathcal{F}_\varepsilon^{-1}g)(\cdot+\varepsilon n)\|^2_{L^2(\varepsilon Q)} = \sum_{n\in\mathbb{Z}^d}\|\widehat{g}_n\|^2_{L^2(\varepsilon Q)} = \int_{\varepsilon Q}\int_{\varepsilon^{-1}Q'}\left|(\mathcal{F}_\varepsilon^{-1}g)(\cdot,\theta)\right|^2 d\theta\,d\mu^\varepsilon,$$

where, for each $z\in\varepsilon Q$,

$$\widehat{g}_n(z) := \left(\frac{\varepsilon}{2\pi}\right)^{d/2}\int_{\varepsilon^{-1}Q'} g(\theta,z)\exp(\mathrm{i}\varepsilon n\cdot\theta)d\theta, \qquad n\in\mathbb{Z}^d,$$

are the Fourier coefficients of the $\varepsilon^{-1}Q'$-periodic function $g(\cdot,z)$. Since the image of $\mathcal{U}_\varepsilon$ contains $C_0^\infty(\mathbb{R}^d)$ and for all $u\in C_0^\infty(\mathbb{R}^d)$ one has $u=\mathcal{U}_\varepsilon\mathcal{F}_\varepsilon u$, it follows that $\mathcal{F}_\varepsilon$ is one-to-one and thus, unitary.

Combining the ε-Floquet transform and the unitary scaling transform

$$\mathcal{T}_\varepsilon h(\theta,y) := \varepsilon^{d/2}h(\theta,\varepsilon y), \qquad \theta\in\varepsilon^{-1}Q',\ y\in Q, \qquad \forall h\in L^2(\varepsilon^{-1}Q'\times\varepsilon Q, d\theta\times d\mu^\varepsilon),$$

$$(\mathcal{T}_\varepsilon^{-1}h)(\theta,z) = \varepsilon^{-d/2}h(\theta,z/\varepsilon), \quad \theta\in\varepsilon^{-1}Q',\ z\in\varepsilon Q, \qquad \forall h\in L^2(\varepsilon^{-1}Q'\times Q, d\theta\times d\mu),$$

we obtain a representation for the operator $\mathcal{A}^\varepsilon$, as follows.

Proposition 2.1.3. *For each $\varepsilon > 0$ the operator $\mathcal{A}^\varepsilon$ is unitarily equivalent to the direct integral of the family $\mathcal{A}_{\varepsilon\theta}$, $\theta \in \varepsilon^{-1}Q'$, namely*

$$(\mathcal{A}^\varepsilon + I)^{-1} = \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}\biggl(\int_{\varepsilon^{-1}Q'}^{\oplus} e_{\varepsilon\theta}(\varepsilon^{-2}\mathcal{A}_{\varepsilon\theta} + I)^{-1}\overline{e_{\varepsilon\theta}}\, d\theta\biggr)\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon,$$

where $\overline{e_{\varepsilon\theta}}$, $e_{\varepsilon\theta}$ represent the operators of multiplication by $\overline{e_{\varepsilon\theta}}$, $e_{\varepsilon\theta}$.

Sketch of proof. The argument is similar to [77]. Taking first solutions $(u, \nabla u) \in H^1(\mathbb{R}^d, d\mu^\varepsilon)$ to (2.1) with $f \in C_0^\infty(\mathbb{R}^d)$, whose both components can be shown to decay exponentially at infinity, *cf.* [77, Proposition 5.3], we denote, for each such u, the "periodic amplitude" of its scaled ε-Floquet transform:

$$u_\theta^\varepsilon := \overline{e_{\varepsilon\theta}}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon u = \biggl(\frac{\varepsilon^2}{2\pi}\biggr)^{d/2}\sum_{n\in Z^d} u(\varepsilon y + \varepsilon n)\exp\bigl(-\mathrm{i}(\varepsilon y + \varepsilon n)\cdot\theta\bigr). \tag{2.8}$$

Note that for any choice of the gradient ∇u (and hence for the one entering (2.1)) the expression

$$\nabla(e_{\varepsilon\theta}u_\theta^\varepsilon)(y) = \varepsilon\biggl(\frac{\varepsilon^2}{2\pi}\biggr)^{d/2}\sum_{n\in Z^d}\nabla u(\varepsilon y + \varepsilon n)\exp\bigl(-\mathrm{i}\varepsilon n\cdot\theta\bigr), \qquad y \in Q,$$

is a gradient of $e_{\varepsilon\theta}u_\theta^\varepsilon$, in the sense that $\bigl(e_{\varepsilon\theta}u_\theta^\varepsilon, \nabla(e_{\varepsilon\theta}u_\theta^\varepsilon)\bigr) \in H^1_{\varepsilon\theta}$, as shown by considering an appropriate sequence $\phi_n \in C_0^\infty(\mathbb{R}^d)$ whose classical gradients converge to ∇u in $L^2(\mathbb{R}^d, d\mu^\varepsilon)$. Therefore

$$\varepsilon^{-2}\int_Q A\nabla(e_{\varepsilon\theta}u_\theta^\varepsilon)\cdot\overline{\nabla(e_{\varepsilon\theta}\varphi)}\, d\mu + \int_Q e_{\varepsilon\theta}u_\theta^\varepsilon\,\overline{e_{\varepsilon\theta}\varphi}\, d\mu = \int_Q e_{\varepsilon\theta}F\,\overline{e_{\varepsilon\theta}\varphi}\, d\mu \quad \forall\,\varphi \in C_\#^\infty, \tag{2.9}$$

where $F = \overline{e_{\varepsilon\theta}}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon f$. The density of $f \in C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, d\mu^\varepsilon)$ implies the claim. $\square$

In what follows we study the asymptotic behaviour of the solutions u_θ^ε to the problems

$$-\varepsilon^{-2}\overline{e_{\varepsilon\theta}}\nabla\cdot A\nabla(e_{\varepsilon\theta}u_\theta^\varepsilon) + u_\theta^\varepsilon = F, \qquad \varepsilon > 0, \quad \theta \in \varepsilon^{-1}Q', \tag{2.10}$$

understood in the sense of the identity (2.9).

2.2 Asymptotic approximation of u_θ^ε

Henceforth we assume that the measure μ is ergodic, *i.e.* whenever $\phi_n \in C_\#^\infty$ and the classical gradients $\nabla\phi_n$ converge to zero in $\bigl[L^2(Q, d\mu)\bigr]^d$, there exists a constant c such that $\phi_n \to c$ in $L^2(Q, d\mu)$. Furthermore, we make the following key assumption on the measure μ.

Assumption 2.2.1. *Suppose that there exists a constant $C_{\mathrm{P}} = C_{\mathrm{P}}(\mu) > 0$ such that for all $\kappa \in Q'$ and $\big(e_\kappa u, \nabla(e_\kappa u)\big) \in H^1_\kappa$ the Poincaré-type inequality*

$$\left\| u - \int_Q u \right\|_{L^2(Q,d\mu)} \leq C_{\mathrm{P}} \big\| \nabla(e_\kappa u) \big\|_{[L^2(Q,d\mu)]^d}. \tag{2.11}$$

In Section 2.2.1 we describe a class of singular measures which satisfy the above assumption.

In what follows we assume that A is a scalar matrix. The analysis of the general case is similar: the modifications required concern the condition on the mean of the unit-cell solutions defined next.

Consider the vector $N = (N_1, N_2, ..., N_d)$ of solutions to the unit cell problems[1]

$$-\nabla \cdot A\nabla N_j = \partial_j A, \qquad \int_Q A N_j = 0, \qquad j = 1, 2, \ldots, d. \tag{2.12}$$

The right-hand side of (2.12) is understood as an element of the space $(H^1_\#)^*$ of linear continuous functionals on $H^1_\#$: for a test function $\varphi \in C^\infty_\#$ the action of $\partial_j A$ on φ is given by

$$\langle \partial_j A, \varphi \rangle = \int_Q A\, \overline{\partial_j \varphi},$$

and the action of the same functional on the whole space $H^1_\#$ is obtained by closure. In particular, for a pair $\mathcal{V} = (v, \nabla v) \in H^1_\#$ we have

$$\langle \partial_j A, \mathcal{V} \rangle = \int_Q A\, \overline{\partial_j v}. \tag{2.13}$$

Proposition 2.2.2. *For each $j = 1, 2, \ldots d$, there exists a unique solution $N_j \in H^1_\#$ to (2.12).*

Proof. It follows from the above assumptions on the measure μ, by setting $\kappa = 0$ in (2.11), that the following Poincaré inequality holds:

$$\left\| u - \int_Q u \right\|_{L^2(Q)} \leq C \|\nabla u\|_{[L^2(Q)]^d}, \quad C > 0, \qquad \forall\ (u, \nabla u) \in H^1_\#. \tag{2.14}$$

[1] In the case of matrix-valued A, the condition on the mean of the solutions N_j, $j = 1, 2, \ldots, d$, is replaced by $\int_Q (A\theta \cdot \theta) N_j = 0$ for $\theta \neq 0$, with no condition imposed for $\theta = 0$, so the mean of N_j (but not its gradient) depends on θ.

Therefore, the sesquilinear form

$$\int_Q A\nabla u \cdot \overline{\nabla v}, \qquad (u, \nabla u), (v, \nabla v) \in H^1_{\#,0},$$

is bounded and coercive, and hence defines an equivalent inner product in $H^1_{\#,0}$. Bearing in mind that (2.13) is a linear bounded functional on $H^1_{\#,0}$, we infer by the Riesz representation theorem (see *e.g.* [7, p. 32]) that for each $j = 1, 2, \dots d$, the equation

$$-\nabla \cdot A\nabla u = \partial_j A,$$

has a unique solution in $\widetilde{N}_j \in H^1_{\#,0}$, and therefore its arbitrary solution in $H^1_{\#}$ has the form $\widetilde{N}_j + a$, $a \in \mathbb{C}$. Setting

$$a = -\left(\int_Q A\right)^{-1} \int_Q A\widetilde{N}_j, \qquad N_j := \widetilde{N_j} + a,$$

concludes the proof. □

Theorem 2.2.3. *Assume that μ is an ergodic measure such that the Poincaré-type inequality* (2.11) *holds, and A is a positive, bounded, Q-periodic, μ-measurable real-valued matrix function. Then, the following estimate holds for the solutions to (2.10) with a constant $C > 0$ independent of ε, θ, F :*

$$\left\| u_\theta^\varepsilon - c_\theta \right\|_{L^2(Q)} \leq C\varepsilon \|F\|_{L^2(Q)}, \tag{2.15}$$

where

$$c_\theta = c_\theta(F) := \left(\theta \cdot \left\{ \int_Q A(\nabla N + I) \right\} \theta + 1\right)^{-1} \int_Q F, \qquad \theta \in \varepsilon^{-1} Q'. \tag{2.16}$$

Corollary 2.2.4. *Under the conditions of the above theorem, there exists $C > 0$ such that*

$$\left\| u^\varepsilon - u^\varepsilon_{\mathrm{hom}} \right\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} \leq C\varepsilon \|f^\varepsilon\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} \quad \forall\, \varepsilon > 0,\ f^\varepsilon \in L^2(\mathbb{R}^d, d\mu^\varepsilon),$$

where u^ε are the solutions to the original problem (2.1) and $u^\varepsilon_{\mathrm{hom}}$ is the solution to the homogenised equation (2.2) with

$$A^{\mathrm{hom}} := \int_Q A(\nabla N + I).$$

Proof of Corollary 2.2.4. In this proof we drop the superscript ε in f^ε for brevity. Consider $f \in L^2(\mathbb{R}^d, d\mu^\varepsilon)$ and denote (see (2.8)) $f_\theta^\varepsilon := \overline{e_{\varepsilon\theta}}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon f$, so that

$$\int_Q f_\theta^\varepsilon = \widehat{f}(\theta,\varepsilon), \quad \theta \in \varepsilon^{-1}Q', \qquad \text{where} \quad \widehat{f}(\theta,\varepsilon) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} f\overline{e_\theta}\,d\mu^\varepsilon, \quad \theta \in \mathbb{R}^d.$$

Also, consider the solutions u_θ^ε to (2.10) with $F = f_\theta^\varepsilon$. Using Proposition 2.1.3 we obtain

$$\begin{aligned} u^\varepsilon - u^\varepsilon_{\rm hom} &= (\mathcal{A}^\varepsilon + I)^{-1}f - (\mathcal{A}^{\rm hom} + I)^{-1}f \\ &= \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}(\varepsilon^{-2}\mathcal{A}_{\varepsilon\theta} + I)^{-1}f_\theta^\varepsilon - (\mathcal{A}^{\rm hom} + I)^{-1}f = \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}u_\theta^\varepsilon - (\mathcal{A}^{\rm hom} + I)^{-1}f \\ &= \bigl\{\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}u_\theta^\varepsilon - \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}c_\theta(f_\theta^\varepsilon)\bigr\} + \bigl\{\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}c_\theta(f_\theta^\varepsilon) - (\mathcal{A}^{\rm hom} + I)^{-1}f\bigr\}, \end{aligned}$$

where the operators $\mathcal{A}^\varepsilon$, $\mathcal{A}^{\rm hom}$ are defined above. In view of Theorem 2.2.3, the unitary property of $\mathcal{F}_\varepsilon$, $\mathcal{T}_\varepsilon$ and the operator of multiplication by $e_{\varepsilon\theta}$, as well as the fact that

$$\begin{aligned} &\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_{\varepsilon\theta}(\theta\cdot A^{\rm hom}\theta + 1)^{-1}\widehat{f}(\theta,\varepsilon) - (2\pi)^{-d/2}\int_{\mathbb{R}^d}(\theta\cdot A^{\rm hom}\theta + 1)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta\,d\theta \\ &= (2\pi)^{-d/2}\biggl\{\int_{\varepsilon^{-1}Q'}(\theta\cdot A^{\rm hom}\theta + 1)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta\,d\theta - \int_{\mathbb{R}^d}(\theta\cdot A^{\rm hom}\theta + 1)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta\,d\theta\biggr\} \\ &= \int_{\mathbb{R}^d\setminus\varepsilon^{-1}Q'}(\theta\cdot A^{\rm hom}\theta + 1)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta\,d\theta, \end{aligned}$$

we obtain

$$\begin{aligned} \bigl\|(\mathcal{A}^\varepsilon + I)^{-1}f - (\mathcal{A}^{\rm hom} + I)^{-1}f\bigr\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} &\le C\varepsilon\|f\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)} + \frac{\varepsilon^2}{\|A^{-1}\|^{-1}\pi^2 + \varepsilon^2}\bigl\|\widehat{f}(\cdot,\varepsilon)\bigr\|_{L^2(\mathbb{R}^d)} \\ &\le \biggl(C\varepsilon + \frac{\varepsilon^2}{\|A^{-1}\|^{-1}\pi^2 + \varepsilon^2}\biggr)\bigl\|f\bigr\|_{L^2(\mathbb{R}^d, d\mu^\varepsilon)}, \end{aligned}$$

from which the claim follows. □

We now proceed to the proof of Theorem 2.2.3. Motivated by the formal asymptotics in powers of ε developed in Section 1.3, we consider the function

$$U_\theta^\varepsilon := c_\theta + \mathrm{i}\varepsilon N_j\theta_j c_\theta + \varepsilon^2 R_\theta^\varepsilon, \tag{2.17}$$

where ∇N_j, $j = 1, 2, \dots, d$, are defined by (2.12), and the "remainder" $R_\theta^\varepsilon \in H^1_\#$ solves

$$\begin{aligned} &-\overline{e_{\varepsilon\theta}}\nabla\cdot A\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon) + \varepsilon^2\int_Q R_\theta^\varepsilon = F + \varepsilon^{-2}\overline{e_{\varepsilon\theta}}\nabla\cdot A\nabla(e_{\varepsilon\theta}c_\theta) + \mathrm{i}\varepsilon^{-1}\overline{e_{\varepsilon\theta}}\nabla\cdot A\nabla(e_{\varepsilon\theta}N_j\theta_j)c_\theta - c_\theta \\ &= F + \mathrm{i}\nabla\cdot\bigl(\mathrm{i}N_j\theta_j A\theta\bigr)c_\theta + \mathrm{i}\theta\cdot A\nabla(\mathrm{i}N_j\theta_j)c_\theta - \mathrm{i}\varepsilon N_j\theta_j\theta\cdot A\theta c_\theta - \theta\cdot A\theta c_\theta - c_\theta =: H_\theta^\varepsilon, \end{aligned} \tag{2.18}$$

where H_θ^ε is an treated as an element of the space $(H^1_\#)^*$. Here for all $\kappa \in Q'$ we set

$$\nabla e_\kappa = \mathrm{i} e_\kappa \kappa, \qquad \nabla(e_\kappa N_j) = e_\kappa(\mathrm{i} N_j \kappa + \nabla N_j), \qquad 1, 2, \dots, d. \tag{2.19}$$

The second equality in (2.18) is verified by taking $\phi \in C^\infty_\#$, noticing that

$$\begin{aligned}
&\Big\langle \varepsilon^{-2} \overline{e_{\varepsilon\theta}} \nabla \cdot A \nabla e_{\varepsilon\theta} + \mathrm{i}\varepsilon^{-1} \overline{e_{\varepsilon\theta}} \nabla \cdot A \nabla (e_{\varepsilon\theta} N_j \theta_j), \varphi \Big\rangle \\
&= -\int_Q \Big(\varepsilon^{-2} A \mathrm{i} e_{\varepsilon\theta} \varepsilon\theta \cdot \overline{\nabla(e_{\varepsilon\theta}\varphi)} + \mathrm{i}\varepsilon^{-1} A e_{\varepsilon\theta} \theta_j (\mathrm{i} N_j \varepsilon\theta + \nabla N_j) \cdot \overline{\nabla(e_{\varepsilon\theta}\varphi)} \Big) \\
&= -\int_Q \Big(\varepsilon^{-2} A \mathrm{i} e_{\varepsilon\theta} \varepsilon\theta \cdot \overline{(\mathrm{i} e_{\varepsilon\theta} \varphi \varepsilon\theta + e_{\varepsilon\theta} \nabla\varphi)} + \mathrm{i}\varepsilon^{-1} A e_{\varepsilon\theta} \theta_j (\mathrm{i} N_j \varepsilon\theta + \nabla N_j) \cdot \overline{(\mathrm{i} e_{\varepsilon\theta} \varphi \varepsilon\theta + e_{\varepsilon\theta} \nabla\varphi)} \Big),
\end{aligned}$$

and finally using (2.12). Note that for c_θ defined by (2.16), the condition $\langle H^\varepsilon_\theta, 1 \rangle = 0$ holds, and in the case $\theta = 0$ the average over Q of the solution R^ε_θ to (2.18) vanishes.

Proposition 2.2.5. *For each $\varepsilon > 0$ and $\theta \in \varepsilon^{-1} Q'$ there exists a unique solution $R^\varepsilon_\theta \in H^1_\#(Q, d\mu)$ for the problem* (2.18).

Proof. The problem (2.18) is understood as

$$\int_Q A \nabla (e_\kappa R^\varepsilon_\theta) \cdot \overline{\nabla(e_\kappa \phi)} + \varepsilon^2 \int_Q R^\varepsilon_\theta \cdot \overline{\int_Q \phi} = \langle H^\varepsilon_\theta, \phi \rangle \quad \forall \phi \in H^1_\#(Q, d\mu).$$

Hence the proof is a result of Lax-Millgram theorem applied to the bilinear form

$$b(u, v) = \int_Q A \nabla (e_\kappa u) \cdot \overline{\nabla(e_\kappa v)} + \varepsilon^2 \int_Q u \cdot \overline{\int_Q v} \quad \forall u, \ v \in H^1_\#(Q, d\mu)$$

indeed the form is bounded and the coercivity follows from the estimate (2.11). $\square$

2.2.1 Discussion of the validity of (2.11) for some singular measures

Consider a finite set $\{\mathcal{P}_j\}_{j=1}^N$ of hyperplanes of dimension d or smaller each of which is parallel some of the Euclidean coordinate axes in $\mathbb{R}^d$ and orthogonal to the complementary coordinate axes and such that $(\cup_{j=1}^N \mathcal{P}_j) \cap Q$ is non-empty and connected.

Define the measure μ on Q by the formula

$$\mu(B) = \Big(\sum_{j=1}^N |\mathcal{P}_j \cap Q|_j \Big)^{-1} \sum_{j=1}^N |\mathcal{P}_j \cap B|_j \ \text{ for all Borel } B \subset Q.$$

where $|\cdot|_j$ represents the d_j-dimensional Lebesgue measure, $d_j = \dim(\mathcal{P}_j)$.

For each $j \in \{1, \dots, N\}$ consider the measure μ_j defined by

$$\mu_j(B) := |\mathcal{P}_j \cap Q|_j^{-1} |\mathcal{P}_j \cap B|_j \ \text{ for all Borel } B \subset Q.$$

Poincaré inequality for a single hyperplane

In this section, we fix $j \in \{1, \dots, N\}$ and assume, without loss of generality, that the plane $\mathcal{P}_j$ passes through zero. We denote by Q_j the d_j-dimensional cross-section of Q by $\mathcal{P}_j$, by $\varkappa_j^{||}$ the vector of those components of the quasimomentum $\varkappa$ that correspond to the selection of the coordinates entering $\mathcal{P}_j$ considered as a subspace of $\mathbb{R}^d$, and by $\varkappa_j^{\perp}$ the vector of those components of $\varkappa$ that do not enter $\varkappa_j^{||}$.

For a function $\phi \in C_{\#}^{\infty}$, at each point $x \in Q$, we decompose the (classical) gradient $\nabla\phi(x)$ into the orthogonal sum of its projection $\nabla_j^{||}\phi(x)$ onto $\mathcal{P}_j$ and its projection $\nabla_j^{\perp}\phi(x)$ onto the orthogonal complement of $\mathcal{P}_j$. We treat $\nabla_j^{||}\phi(x)$ and $\nabla_j^{\perp}\phi(x)$ as elements of $\mathbb{R}^{d_j}$ and $\mathbb{R}^{d-d_j}$, respectively. Clearly, for each $\varkappa \in Q'$, one has, pointwise in Q,

$$\bigl|\nabla(e_\kappa\phi)\bigr|^2 = \bigl|\mathrm{i}\phi\varkappa + \nabla\phi\bigr|^2 = \bigl|\mathrm{i}\phi\varkappa_j^{||} + \nabla^{||}\phi\bigr|^2 + \bigl|\mathrm{i}\phi\varkappa_j^{\perp} + \nabla^{\perp}\phi\bigr|^2 \geq \bigl|\mathrm{i}\phi\varkappa_j^{||} + \nabla^{||}\phi\bigr|^2, \quad (2.20)$$

where the norms are considered in appropriate Euclidean spaces.

Next, we write

$$\phi(\widetilde{x}) - \int_Q \phi d\mu_j = \sum_{l \in \mathbb{Z}^{d_j} \setminus \{0\}} c_l \exp(2\pi\mathrm{i}l \cdot \widetilde{x}), \quad \widetilde{x} \in Q_j, \qquad c_l \in \mathbb{C},\ l \in \mathbb{Z}^{d_j},$$

and notice that for all j one has, assuming ϕ is non-constant on $\mathcal{P}_j \cap Q$,

$$\biggl(\int_Q \biggl|\phi - \int_Q \phi d\mu_j\biggr|^2 d\mu_j\biggr)^{-1} \int_Q \bigl|\mathrm{i}\phi\varkappa_j^{||} + \nabla^{||}\phi\bigr|^2 d\mu_j$$
$$= \biggl(\sum_{l,m \in \mathbb{Z}^{d_j} \setminus \{0\}} \alpha_{lm} c_l \overline{c_m}\biggr)^{-1} \biggl(\sum_{l,m \in \mathbb{Z}^{d_j} \setminus \{0\}} \alpha_{lm} c_l \overline{c_m} (\varkappa_j^{||} + 2\pi l) \cdot (\varkappa_j^{||} + 2\pi m)\biggr),$$

where

$$\alpha_{lm} := \int_{Q_j} \exp\bigl(2\pi\mathrm{i}(l-m) \cdot \widetilde{x}\bigr) d\mu_j(\widetilde{x}) = \begin{cases} 1, & l = m, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\biggl(\int_Q\Bigl|\phi-\int_Q\phi d\mu_j\Bigr|^2 d\mu_j\biggr)^{-1}\int_Q\bigl|\mathrm{i}\phi\varkappa_j^{||}+\nabla^{||}\phi\bigr|^2 d\mu_j$$
$$=\biggl(\sum_{l\in\mathbb{Z}^{d_j}\setminus\{0\}}|c_l|^2\biggr)^{-1}\biggl(\sum_{l,m\in\mathbb{Z}^{d_j}\setminus\{0\}}|c_l|^2|\varkappa_j^{||}+2\pi l|^2\biggr)\geq\pi^2,$$

or equivalently

$$\int_Q\Bigl|\phi-\int_Q\phi d\mu_j\Bigr|^2 d\mu_j\leq\pi^{-2}\int_Q\bigl|\mathrm{i}\phi\varkappa_j^{||}+\nabla^{||}\phi\bigr|^2 d\mu_j \tag{2.21}$$

If the function ϕ is constant on $\mathcal{P}_j\cap Q$, the inequality (2.21) is satisfied trivially.

Connectivity argument

For the measure $\mu=\sum_{j=1}^N\mu_j$ and $\phi\in C_\#^\infty$, we denote by $\nabla^{||}(e_\kappa\phi)$ the tangential gradient of ϕ at points of $\mathrm{supp}(\mu)$, *i.e.* the orthogonal projection of $\nabla(e_\kappa\phi)$ onto $\mathrm{supp}(\mu)$.

Suppose that for $j,k\in\{1,\dots,N\}$ the hyperplanes $\mathcal{P}_j$ and $\mathcal{P}_k$ intersect and fix a point $\alpha_{jk}\in\mathcal{P}_j\cap\mathcal{P}_k\cap Q$. For any $\kappa\in Q'$, any function $\phi\in C_\#^\infty$, and all $x\in\mathcal{P}_j\cap Q$, $y\in\mathcal{P}_k\cap Q$, one has

$$\begin{aligned}e_\kappa(x)\phi(x)-e_\kappa(y)\phi(y)&=\int_{\alpha_{jk}}^x\nabla(e_\kappa\phi)\bigl(\alpha_{jk}+t(x-\alpha_{jk})\bigr)dt\cdot(x-\alpha_{jk})\\&\quad-\int_{\alpha_{jk}}^y\nabla(e_\kappa\phi)\bigl(\alpha_{jk}+t(y-\alpha_{jk})\bigr)dt\cdot(y-\alpha_{jk}).\end{aligned}\tag{2.22}$$

Multiplying both sides of (2.22) by $e_\kappa(y)^{-1}=e_\kappa(-y)$ and integrating over $y\in Q$ with respect to the measure μ_k (recalling that $\mathrm{supp}(\mu_k)=\mathcal{P}_k\cap Q$) yields

$$e_\kappa(x)\phi(x)\int_Q e_\kappa^{-1}d\mu_k-\int_Q\phi d\mu_k\leq\sqrt{2}\Bigl(\bigl\|\nabla^{||}(e_\kappa\phi)\bigr\|_{L^2(Q,d\mu_j)}+\bigl\|\nabla^{||}(e_\kappa\phi)\bigr\|_{L^2(Q,d\mu_k)}\Bigr)\quad\forall x\in\mathcal{P}_j\cap Q. \tag{2.23}$$

Furthermore, multiplying both sides of (2.23) by $e_\kappa(x)^{-1}$ and integrating over $x\in Q$ with respect to the measure μ_j yields

$$\int_Q\phi d\mu_j-\int_Q\phi d\mu_k\leq\sqrt{2}\Bigl(\bigl\|\nabla^{||}(e_\kappa\phi)\bigr\|_{L^2(Q,d\mu_j)}+\bigl\|\nabla^{||}(e_\kappa\phi)\bigr\|_{L^2(Q,d\mu_k)}\Bigr)\leq\sqrt{2}\|\nabla^{||}(e_\kappa\phi)\bigr\|_{L^2(Q,d\mu)}.$$

By interchanging k and j if necessary, we thus obtain

$$\left|\int_Q \phi d\mu_j - \int_Q \phi d\mu_k\right| \le \sqrt{2}\big\|\nabla^{||}(e_\kappa\phi)\big\|_{L^2(Q,d\mu)}.$$

Next, notice that since $(\cup_{j=1}^N \mathcal{P}_j)\cap Q$ is connected by assumption, for each pair of planes in the union there is a "path" from one plane to the other involving at most N planes, such that any "adjacent" planes in the path intersect. It follows that for all pairs j,k the bound

$$\left|\int_Q \phi d\mu_j - \int_Q \phi d\mu_k\right| \le \sqrt{2}N\big\|\nabla^{||}(e_\kappa\phi)\big\|_{L^2(Q,d\mu)}. \tag{2.24}$$

holds.

Finally, using (2.24) and standard arithmetic inequalities, we obtain

$$\begin{aligned}
\int_Q\left|\phi - \int_Q\phi\right|^2 d\mu &= \sum_{j=1}^N\int_Q\left|\phi - \int_Q\phi\right|^2 d\mu_j = \sum_{j=1}^N\int_Q\left|\phi - \sum_{k=1}^N N^{-1}\int_Q\phi d\mu_k\right|^2 d\mu_j\\
&= \sum_{j=1}^N\int_Q\left|\sum_{k=1}^N N^{-1}\Big(\phi - \int_Q\phi d\mu_k\Big)\right|^2 d\mu_j \le \sum_{j=1}^N\sum_{k=1}^N N^{-1}\int_Q\left|\Big(\phi - \int_Q\phi d\mu_k\Big)\right|^2 d\mu_j\\
&= \sum_{j=1}^N\sum_{k=1}^N N^{-1}\int_Q\left|\phi - \int_Q\phi d\mu_j + \Big(\int_Q\phi d\mu_j - \int_Q\phi d\mu_k\Big)\right|^2 d\mu_j\\
&\le 2\sum_{j=1}^N\sum_{k=1}^N N^{-1}\left\{\int_Q\left|\phi - \int_Q\phi d\mu_j\right|^2 d\mu_j + 2N^2\|\nabla^{||}(e_\kappa\phi)\|^2_{L^2(Q,d\mu)}\right\}\\
&= 2\sum_{j=1}^N\left\{\int_Q\left|\phi - \int_Q\phi d\mu_j\right|^2 d\mu_j + 2N^2\|\nabla^{||}(e_\kappa\phi)\|^2_{L^2(Q,d\mu)}\right\}\\
&\le 2\sum_{j=1}^N\left\{\pi^{-2}\|\nabla^{||}(e_\kappa\phi)\|^2_{L^2(Q,d\mu_j)} + 2N^2\|\nabla^{||}(e_\kappa\phi)\|^2_{L^2(Q,d\mu)}\right\}\\
&\le 2\big(\pi^{-2} + 2N^3\big)\|\nabla^{||}(e_\kappa\phi)\|^2_{L^2(Q,d\mu)}.
\end{aligned}$$

Combining the above bound with (2.20), where we notice that for each $j = 1,\dots,N$, on $\operatorname{supp}(\mu_j)$ one has

$$\nabla^{||}(e_\kappa\phi) = e_\kappa(\mathrm{i}\phi\varkappa_j^{||} + \nabla^{||}\phi),$$

we obtain

$$\int_Q\left|\phi - \int_Q\phi\right|^2 d\mu \le C_{\mathrm{P}}\big\|\nabla(e_\kappa\phi)\big\|^2_{L^2(Q,d\mu)}, \tag{2.25}$$

with

$$C_{\rm P} = 2\big(\pi^{-2} + 2N^3\big). \tag{2.26}$$

Finally, approximating an arbitrary $\big(e_\varkappa u, \nabla(e_\varkappa u)\big) \in H^1_\varkappa$ by pairs $\big(e_\varkappa \phi, \nabla(e_\varkappa \phi)\big)$, $\phi \in C^\infty_\#$, in line with Definition 2.1.1, and passing to the limit as $n \to \infty$ in the bound (2.25) yields the inequality (2.11), with $C_{\rm P}$ given by (2.26).

2.3 Estimate for the "remainder" $\varepsilon^2 R^\varepsilon_\theta$

Theorem 2.3.1. *Suppose that $\theta \neq 0$, $\varepsilon > 0$. For the solution R^ε_θ to the problem (2.18) the following estimates hold with $C > 0$:*

$$\biggl\| R^\varepsilon_\theta - \int_Q R^\varepsilon_\theta \biggr\|_{L^2(Q)} \le C \|F\|_{L^2(Q)}, \qquad \biggl| \int_Q R^\varepsilon_\theta \biggr| \le C\varepsilon^{-1} \|F\|_{L^2(Q)}. \tag{2.27}$$

Proof. Consider a sequence of functions $\phi_n \in C^\infty_\#$ that converges in $L^2(Q)$ to R^ε_θ, such that

$$\nabla(e_{\varepsilon\theta}\phi_n) \overset{[L^2(Q)]^d}{\longrightarrow} \nabla\big(e_{\varepsilon\theta} R^\varepsilon_\theta\big),$$

and, equivalently,

$$\nabla\biggl[e_{\varepsilon\theta}\biggl(\phi_n - \int_Q R^\varepsilon_\theta\biggr)\biggr] \overset{[L^2(Q)]^d}{\longrightarrow} \nabla\biggl[e_{\varepsilon\theta}\biggl(R^\varepsilon_\theta - \int_Q R^\varepsilon_\theta\biggr)\biggr].$$

It follows from (2.18) that

$$\int_Q A\nabla(e_{\varepsilon\theta}R^\varepsilon_\theta)\cdot\overline{\nabla(e_{\varepsilon\theta}\phi_n)} + \varepsilon^2\int_Q R^\varepsilon_\theta \overline{\int_Q \phi_n} = \langle H^\varepsilon_\theta, 1\rangle \overline{\int_Q R^\varepsilon_\theta} + \biggl\langle H^\varepsilon_\theta, \phi_n - \int_Q R^\varepsilon_\theta\biggr\rangle.$$

Furthermore, using the fact that $\langle H^\varepsilon_\theta, 1\rangle = 0$ and the formula

$$\nabla\phi_n = \overline{e_{\varepsilon\theta}}\biggl\{\nabla\biggl[e_{\varepsilon\theta}\biggl(\phi_n - \int_Q R^\varepsilon_\theta\biggr)\biggr] - \biggl(\phi_n - \int_Q R^\varepsilon_\theta\biggr)\nabla e_{\varepsilon\theta}\biggr\},$$

where all the gradients are understood in the classical sense, we obtain

$$\int_Q A\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)\cdot\overline{\nabla(e_{\varepsilon\theta}\phi_n)}+\varepsilon^2\int_Q R_\theta^\varepsilon\overline{\int_Q\phi_n}=\int_Q\big(F+\mathrm{i}\theta\cdot A\nabla(\mathrm{i}N_j\theta_j)c_\theta$$
$$-\mathrm{i}\varepsilon N_j\theta_j\theta\cdot A\theta c_\theta-\theta\cdot A\theta c_\theta-c_\theta\big)\overline{\Big(\phi_n-\int_Q R_\theta^\varepsilon\Big)}$$
$$+c_\theta\int_Q e_{\varepsilon\theta}N_j\theta_jA\theta\cdot\overline{\bigg\{\nabla\Big[e_{\varepsilon\theta}\Big(\phi_n-\int_Q R_\theta^\varepsilon\Big)\Big]-\Big(\phi_n-\int_Q R_\theta^\varepsilon\Big)\nabla e_{\varepsilon\theta}\bigg\}}. \tag{2.28}$$

Passing to the limit as $n\to\infty$ yields

$$\int_Q A\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)\cdot\overline{\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)}+\varepsilon^2\bigg|\int_Q R_\theta^\varepsilon\bigg|^2=\int_Q\Big[F-c_\theta\Big(\theta\cdot A\nabla(N_j\theta_j)+(\mathrm{i}\varepsilon N_j\theta_j+1)\theta\cdot A\theta$$
$$+e_{\varepsilon\theta}N_j\theta_j\theta\cdot\overline{A\nabla e_{\varepsilon\theta}}+1\Big)\Big]\overline{\Big(R_\theta^\varepsilon-\int_Q R_\theta^\varepsilon\Big)}+c_\theta\int_Q e_{\varepsilon\theta}N_j\theta_j\theta\cdot\overline{A\nabla\Big\{e_{\varepsilon\theta}\Big(R_\theta^\varepsilon-\int_Q R_\theta^\varepsilon\Big)\Big\}}. \tag{2.29}$$

Consider the solution $\Phi_\theta^\varepsilon\in H^1_\#$ to the problem

$$-\overline{e_{\varepsilon\theta}}\nabla\cdot A\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon)+\varepsilon^2\int_Q\Phi_\theta^\varepsilon=-\overline{e_{\varepsilon\theta}}\nabla\cdot\big(e_{\varepsilon\theta}N_j\theta_jA\theta\big)c_\theta, \tag{2.30}$$

so that for the last term in (2.29) we obtain

$$c_\theta\int_Q e_{\varepsilon\theta}N_j\theta_j\theta\cdot\overline{A\nabla\Big\{e_{\varepsilon\theta}\Big(R_\theta^\varepsilon-\int_Q R_\theta^\varepsilon\Big)\Big\}}=\int_Q A\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon)\cdot\overline{\nabla\Big\{e_{\varepsilon\theta}\Big(R_\theta^\varepsilon-\int_Q R_\theta^\varepsilon\Big)\Big\}}. \tag{2.31}$$

In what follows we use the uniform estimate

$$\big\|\sqrt{A}\nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon)\big\|_{[L^2(Q)]^d}\le C\|F\|_{L^2(Q)}, \tag{2.32}$$

which is obtained by using Φ_θ^ε as a test function in the integral formulation of (2.30).

We would like to rewrite the expression on the right-hand side of (2.31) using Φ_θ^ε as a test function in the integral identity for (2.18). Recall that the gradient of an arbitrary function in $H^1_\#$, for a general measure μ, is not defined in a unique way. However, for the solution Φ_θ^ε to (2.30) there exists a natural choice of the gradient $\nabla\Phi_\theta^\varepsilon$, dictated by

(2.30). Indeed, consider sequences ϕ_n, $\psi_n \in C^\infty_\#$ converging to Φ^ε_θ in $L^2(Q)$ so that

$$\nabla(e_{\varepsilon\theta}\phi_n) \overset{[L^2(Q)]^d}{\longrightarrow} \nabla\bigl(e_{\varepsilon\theta}\Phi^\varepsilon_\theta\bigr), \qquad \nabla(e_{\varepsilon\theta}\psi_n) \overset{[L^2(Q)]^d}{\longrightarrow} \nabla\bigl(e_{\varepsilon\theta}\Phi^\varepsilon_\theta\bigr).$$

Clearly, the difference $\nabla(e_{\varepsilon\theta}\phi_n)-\nabla(e_{\varepsilon\theta}\psi_n)$ converges to zero, and hence so does $\nabla\phi_n - \nabla\psi_n$. In what follows we denote by $\nabla\Phi^\varepsilon_\theta$ the common L^2-limit of gradients $\nabla\phi_n$ for sequences $\phi_n \in C^\infty_\#$ with the above properties. Passing to the limit, as $n\to\infty$, in the identity $\nabla\phi_n = \overline{e_{\varepsilon\theta}}\bigl(\nabla(e_{\varepsilon\theta}\phi_n) - \mathrm{i}\varepsilon\phi_n\theta\bigr)$, we obtain the formula

$$\nabla\Phi^\varepsilon_\theta = \overline{e_{\varepsilon\theta}}\bigl(\nabla(e_{\varepsilon\theta}\Phi^\varepsilon_\theta) - \mathrm{i}\varepsilon\Phi^\varepsilon_\theta\theta\bigr). \tag{2.33}$$

The unique choice of $\nabla\Phi^\varepsilon_\theta$, as above, allows us to write

$$\int_Q A\nabla(e_{\varepsilon\theta}R^\varepsilon_\theta)\cdot\overline{\nabla(e_{\varepsilon\theta}\Phi^\varepsilon_\theta)} + \varepsilon^2\int_Q R^\varepsilon_\theta\overline{\int_Q \Phi^\varepsilon_\theta} = \langle H^\varepsilon_\theta, \Phi^\varepsilon_\theta\rangle \equiv \Bigl\langle H^\varepsilon_\theta, \Phi^\varepsilon_\theta - \int_Q \Phi^\varepsilon_\theta\Bigr\rangle,$$

so that

$$\begin{aligned}\int_Q A\nabla(e_{\varepsilon\theta}\Phi^\varepsilon_\theta)\cdot\overline{\nabla\Bigl\{e_{\varepsilon\theta}\Bigl(R^\varepsilon_\theta - \int_Q R^\varepsilon_\theta\Bigr)\Bigr\}} &= \overline{\Bigl\langle H^\varepsilon_\theta, \Phi^\varepsilon_\theta - \int_Q \Phi^\varepsilon_\theta\Bigr\rangle}\\ -\overline{\int_Q R^\varepsilon_\theta}\Bigl(\int_Q A\nabla(e_{\varepsilon\theta}\Phi^\varepsilon_\theta)\cdot\overline{\nabla e_{\varepsilon\theta}} + \varepsilon^2\int_Q \Phi^\varepsilon_\theta\Bigr) &= \overline{\Bigl\langle H^\varepsilon_\theta, \Phi^\varepsilon_\theta - \int_Q \Phi^\varepsilon_\theta\Bigr\rangle},\end{aligned} \tag{2.34}$$

where the values of the functional H^ε_θ are chosen accordingly. In the last equality in (2.34) we use the fact that

$$\int_Q A\nabla(e_{\varepsilon\theta}\Phi^\varepsilon_\theta)\cdot\overline{\nabla e_{\varepsilon\theta}} + \varepsilon^2\int_Q \Phi^\varepsilon_\theta = 0,$$

by setting the unity as a test function in the integral formulation of (2.30) and recalling that (*cf.* (2.12))

$$\int_Q AN_j = 0, \qquad j = 1, 2, \dots, d.$$

Combining (2.29), (2.31) and (2.34) yields

$$\begin{aligned}\int_Q A\nabla(e_{\varepsilon\theta}R^\varepsilon_\theta)\cdot\overline{\nabla(e_{\varepsilon\theta}R^\varepsilon_\theta)} + \varepsilon^2\Bigl|\int_Q R^\varepsilon_\theta\Bigr|^2 &= \int_Q\Bigl[F - c_\theta\Bigl(\theta\cdot A\nabla(N_j\theta_j) + (\mathrm{i}\varepsilon N_j\theta_j + 1)\theta\cdot A\theta\\ &+ e_{\varepsilon\theta}N_j\theta_j\theta\cdot\overline{A\nabla e_{\varepsilon\theta}} + 1\Bigr)\Bigr]\overline{\Bigl(R^\varepsilon_\theta - \int_Q R^\varepsilon_\theta\Bigr)} + \overline{\Bigl\langle H^\varepsilon_\theta, \Phi^\varepsilon_\theta - \int_Q \Phi^\varepsilon_\theta\Bigr\rangle}.\end{aligned} \tag{2.35}$$

Lemma 2.3.2. *The last term on the right-hand side of (2.35) is bounded, uniformly in* ε, θ :

$$\left|\left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle\right| \leq C\|F\|_{L^2(Q)}, \qquad C > 0.$$

Proof. Notice that

$$\left\langle H_\theta^\varepsilon, \Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon \right\rangle = \int_Q \Bigl(F + \mathrm{i}\nabla \cdot \bigl(\mathrm{i}N_j\theta_j A\theta\bigr)c_\theta + \mathrm{i}\theta \cdot A\nabla(\mathrm{i}N_j\theta_j)c_\theta$$
$$- \mathrm{i}\varepsilon N_j\theta_j\theta \cdot A\theta c_\theta - \theta \cdot A\theta c_\theta - c_\theta\Bigr)\left(\Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon\right) + c_\theta \int_Q N_j\theta_j A\theta \cdot \nabla\Phi_\theta^\varepsilon, \tag{2.36}$$

where the second term is re-written using (2.33):

$$\int_Q N_j\theta_j A\theta \cdot \nabla\Phi_\theta^\varepsilon = \int_Q \overline{e_{\varepsilon\theta}} N_j\theta_j A\theta \cdot \nabla(e_{\varepsilon\theta}\Phi_\theta^\varepsilon) - \mathrm{i}\varepsilon \int_Q N_j\theta_j A\theta \cdot \theta\left(\Phi_\theta^\varepsilon - \int_Q \Phi_\theta^\varepsilon\right).$$

Applying the Hölder inequality to both terms on the right-hand side of (2.36), using the Poincaré inequality (2.11) for Φ_θ^ε, and taking into the account the bound (2.32) yields the required estimate. □

Combining the above lemma, the Poincaré inequality (2.11) for R_θ^ε and Hölder inequality for the first term on the right-hand side of (2.35), we obtain the uniform bound

$$\bigl\|\sqrt{A}\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)\bigr\|_{[L^2(Q)]^d} \leq C\|F\|_{L^2(Q)}. \tag{2.37}$$

Finally, the bound (2.37) combined with (2.11) implies the first estimate in (2.27), whereas the same bound and equation (2.35) implies the second estimate in (2.27). This completes the proof of the theorem. □

Note that in the case $\theta = 0$ the equality (2.29) takes the form

$$\int_Q A\nabla R_0^\varepsilon \cdot \overline{\nabla R_0^\varepsilon} + \varepsilon^2\left|\int_Q R_0^\varepsilon\right|^2 = \int_Q F\overline{R_0^\varepsilon}, \tag{2.38}$$

and taking into account (2.11) with $\kappa = 0$, we obtain

$$\bigl\|\nabla R_0^\varepsilon\bigr\|_{[L^2(Q)]^d} \leq C\|F\|_{L^2(Q)}.$$

The last estimate implies the first bound in (2.27) by (2.11) with $\kappa = 0$ and the second bound in (2.27) by using (2.38) once again.

Corollary 2.3.3. *The following estimate holds uniformly in $\varepsilon > 0$, $\theta \in \varepsilon^{-1}Q'$, $F \in L^2(Q)$:*

$$\|U_\theta^\varepsilon - c_\theta\|_{L^2(Q)} \leq C\varepsilon \|F\|_{L^2(Q)}.$$

2.4 Conclusion of the convergence estimate

Here we estimate the error incurred by using the approximation U_θ^ε in (2.10).

Proposition 2.4.1. *The difference $z_\theta^\varepsilon := u_\theta^\varepsilon - U_\theta^\varepsilon$ satisfies the estimate*

$$\|z_\theta^\varepsilon\|_{L^2(Q)} \leq C\varepsilon \|F\|_{L^2(Q)}, \quad C > 0, \qquad \forall\, \varepsilon > 0,\ \theta \in \varepsilon^{-1}Q',\ F \in L^2(Q).$$

Proof. It follows from (2.10), (2.17), (2.16), (2.12), (2.18), by a direct calculation, that

$$-\varepsilon^{-2}\overline{e_{\varepsilon\theta}}\nabla \cdot A\nabla(e_{\varepsilon\theta} z_\theta^\varepsilon) + z_\theta^\varepsilon = -\mathrm{i}\varepsilon N_j \theta_j c_\theta - \varepsilon^2 \biggl(R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon\biggr). \tag{2.39}$$

In particular, using z_θ^ε as a test function in (2.39), we obtain

$$\varepsilon^{-2}\int A\nabla(e_{\varepsilon\theta} z_\theta^\varepsilon) \cdot \overline{\nabla e_{\varepsilon\theta} z_\theta^\varepsilon} + \int_Q |z_\theta^\varepsilon|^2 = -\mathrm{i}\varepsilon c_\theta \theta_j \int_Q N_j \overline{z_\theta^\varepsilon} - \varepsilon^2 \int_Q \biggl(R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon\biggr)\overline{z_\theta^\varepsilon},$$

and hence

$$\begin{aligned}\|z_\theta^\varepsilon\|_{L^2(Q)}^2 &\leq \varepsilon |c_\theta||\theta|\|N\|_{[L^2(Q)]^d}\|z_\theta^\varepsilon\|_{L^2(Q)} + \varepsilon^2 C_{\mathrm{P}} \bigl\|\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)\bigr\|_{[L^2(Q)]^d}\|z_\theta^\varepsilon\|_{L^2(Q)}\\ &\leq \varepsilon\Bigl(|c_\theta||\theta|\|N\|_{[L^2(Q)]^d} + \varepsilon C\bigl\|\sqrt{A}\nabla(e_{\varepsilon\theta}R_\theta^\varepsilon)\bigr\|_{[L^2(Q)]^d}\Bigr)\|z_\theta^\varepsilon\|_{L^2(Q)},\end{aligned}$$

where we use the inequality (2.11) once again and the fact that A is uniformly positive. The claim follows, by virtue of the formula (2.16) and the estimate (2.37). □

Combining Corollary 2.3.3 and Proposition 2.4.1 concludes the proof of Theorem 2.2.3.

Chapter 3

System of Maxwell equations in functions spaces with respect to Borel measures

Introduction

In this chapter we proceed to the analysis of a vectorial problem, in particular we introduce the system of Maxwell equations on singular periodic structures and carry out an initial analysis of it. The structure of the present chapter is linked to Chapter 1. In fact, here we tackle the analysis of a vectorial problem for the system of Maxwell equations and we understand, through two classical methods in homogenisation, the structure of the leading order term in two-scale asymptotic and its relation to the Helmholtz decomposition. This approach is analogous to the one adopted for the scalar elliptic problem (1.7) in the first chapter.

As was mentioned in the Introduction, the homogenisation problem for the Maxwell system has been studied intensively in the classical case of Lebesgue measure setting. In particular, in the books [74] by Zhikov, Kozlov and Oleinik, and [6] by Bensoussan, Lions and Papanicolaou it is carried out with the method of compensated compactness, and in the work by Wellander [69] with the two-scale convergence. Operator-norm estimates for the system of Maxwell equations were obtained in the setting of a whole space with Lebesgue measure by Birman and Suslina in [8, Chapter 7], [10], and by Suslina in [58], [60]. These estimates are a consequence of a theoretical approach based on spectral perturbation theory and developed in [8], [11].

The study of the system of Maxwell equations in the setting of periodic structures, represented by an arbitrary (periodic) Borel measure μ, will cover both cases of measures that are absolutely continuous with respect to the Lebesgue measure (including the classical case of the Lebesgue measure itself), and measures that are singular. The latter can be viewed as limits of "thin structures", involving elements whose thickness is much smaller than length. The Maxwell equations describe electromagnetic phenomena in a domain $\Omega \in \mathbb{R}^3$ occupied by a medium in times $t \in \mathbb{R}$. The formulation we use in the present thesis (see for example the classical books by Cessenat [17] and Jackson [35]) is

$$\begin{cases} \operatorname{curl} \mathcal{H} = \partial_t \mathcal{D} + \mathcal{J}, \\ \operatorname{curl} \mathcal{E} = -\partial_t \mathcal{B}, \\ \operatorname{div} \mathcal{D} = \rho, \quad \operatorname{div} \mathcal{B} = 0, \end{cases} \tag{3.1}$$

where $\mathcal{E}$ is the electric field, $\mathcal{H}$ is the magnetic field, $\mathcal{D}$ is the electric displacement and $\mathcal{B}$ is the magnetic induction. $\mathcal{J}$ is the current density and ρ the charge density. In the present work we analyse the case with $\rho = 0$.
$\mathcal{E}$, $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{H}$ are linked from the constitutive relations

$$\mathcal{D} = \hat{\eta} *_t \mathcal{E}, \quad \mathcal{B} = \hat{\nu} *_t \mathcal{H}. \tag{3.2}$$

With $*_t$ we label the convolution with respect the time t. Here $\hat{\eta}(x,t)$ is the dielectric permittivity and $\hat{\nu}(x,t)$ is the magnetic permeability. We assume these two functions separable in the two variables, so $\hat{\eta}(x,t) = \eta(x)\delta(t)$ and $\hat{\nu}(x,t) = \nu(x)\delta(t)$. The Maxwell system in the zero charge density case, is

$$\begin{cases} \operatorname{curl} \mathcal{H} = \eta \partial_t \mathcal{E} + \mathcal{J}, \\ \operatorname{curl} \mathcal{E} = -\nu \partial_t \mathcal{H}, \\ \operatorname{div} \mathcal{E} = 0, \quad \operatorname{div} \mathcal{H} = 0. \end{cases}$$

Our analysis is about the harmonic in time Maxwell system, thus we consider

$$\mathcal{D} = De^{i\omega t}, \quad \mathcal{B} = Be^{i\omega t}, \quad \mathcal{E} = Ee^{i\omega t}, \quad \mathcal{H} = He^{i\omega t},$$

where ω is the propagation frequency. The Maxwell system in the time-harmonic case with zero charge density is

$$\begin{cases} \eta^{-1} \operatorname{curl} H = i\omega E + \eta^{-1} J, \\ \nu^{-1} \operatorname{curl} E = -i\omega H, \\ \operatorname{div} E = 0, \quad \operatorname{div} H = 0, \end{cases} \tag{3.3}$$

where η is the electric permittivity and ν the magnetic permeability.

In order to study the system of Maxwell equations from the mathematical point of view, we need to write it in a dimensionless way. Following the idea developed in [5, Chapter 1] we define

$$H = \phi \widetilde{H}, \quad E = \psi \widetilde{E}, \tag{3.4}$$

where ϕ, ψ are some fixed quantities with the dimensions of the magnetic and electric field, and $\widetilde{H}$, $\widetilde{E}$ are the corresponding dimensionless quantities representing the magnetic and electric fields. Starting from (3.3) it is sufficient to do the dimensional analysis for the homogeneous system. Using (3.4), the first line of the homogeneous formulation of (3.3) becomes

$$\eta^{-1}\frac{\phi}{\psi}\operatorname{curl}\widetilde{H} = i\omega\widetilde{E}. \tag{3.5}$$

Note that $\frac{\phi}{\psi} = \sqrt{\frac{\eta_0}{\nu_0}}$, where η_0 and ν_0 are the electric permittivity and the magnetic permeability in vacuum. Multiplying both sides of (3.5) by $\sqrt{\eta_0}$ on has

$$\left(\frac{\eta}{\eta_0}\right)^{-1}\operatorname{curl}\widetilde{H} = i\omega\sqrt{\nu_0\eta_0}\widetilde{E}.$$

It follows that the dimensional first equation of (3.3) can be written as

$$\left(\frac{\eta}{\eta_0}\right)^{-1}(x/d)\operatorname{curl}_x\widetilde{H} = i\frac{\omega}{\omega_0}\omega_0\sqrt{\nu_0\eta_0}\widetilde{E}, \tag{3.6}$$

where d is the period, ω is the propagation frequency in the medium and ω_0 is the propagation frequency in the vacuum. Note that

$$\omega_0 = \frac{2\pi c_0}{\lambda_0},$$

where c_0 is the wavespeed in vacuum and λ_0 the wavelength in vacuum. Furthermore, note that the quantity ω/ω_0 is dimensionless.

Introduce the non-dimensional parameter $\widetilde{x} = 2\pi x/\lambda_0$, so (3.6) becomes

$$\frac{2\pi}{\lambda_0}\left(\frac{\eta}{\eta_0}\right)^{-1}\left(\frac{\widetilde{x}}{2\pi d/\lambda_0}\right)\operatorname{curl}_{\widetilde{x}}\widetilde{H} = i\frac{\omega}{\omega_0}\frac{2\pi c_0}{\lambda_0}\sqrt{\nu_0\eta_0}\widetilde{E}.$$

Labelling $\varepsilon := 2\pi d/\lambda_0$, on has

$$\left(\frac{\eta}{\eta_0}\right)^{-1}(\widetilde{x}/\varepsilon)\operatorname{curl}_{\widetilde{x}}\widetilde{H} = i\frac{\omega}{\omega_0}c_0\sqrt{\nu_0\eta_0}\widetilde{E},$$

where $c_0\sqrt{\nu_0\eta_0} = 1$. Now scaling the problem with $y = \widetilde{x}/\varepsilon$, on has

$$\varepsilon^{-1}\left(\frac{\eta}{\eta_0}\right)^{-1}(y)\operatorname{curl}_y\widetilde{H} = i\frac{\omega}{\omega_0}\widetilde{E},$$

equivalently

$$\left(\frac{\eta}{\eta_0}\right)^{-1}(y)\operatorname{curl}_y\widetilde{H} = i\frac{\omega}{\omega_0}\frac{\lambda_0}{2\pi d}\widetilde{E}. \tag{3.7}$$

In an analogous way we do the dimensional analysis for the second equation in (3.3), and we obtain

$$\left(\frac{\nu}{\nu_0}\right)^{-1}(y)\,\mathrm{curl}_y\,\widetilde{E} = -i\frac{\omega}{\omega_0}\frac{\lambda_0}{2\pi d}\widetilde{H}. \tag{3.8}$$

The dimensional analysis includes the constitutive relations as well. Define

$$D = \xi\widetilde{D}, \quad B = \zeta\widetilde{B}, \tag{3.9}$$

where ξ, ζ are some fixed quantities with the dimensions of the electric displacement and the magnetic induction, and $\widetilde{D}$, $\widetilde{B}$ are the corresponding dimensionless quantities representing the electric displacement and the magnetic induction. Using (3.4) and (3.9) in the constitutive relations, and noting that $\frac{\xi}{\psi} = \eta_0$ and $\frac{\zeta}{\phi} = \nu_0$, one has

$$\widetilde{D} = \frac{\eta}{\eta_0}\widetilde{E}, \quad \widetilde{B} = \frac{\nu}{\nu_0}\widetilde{H}.$$

It follows that we can write the dimensionless system of Maxwell equations:

$$\begin{pmatrix} 0 & -A\,\mathrm{curl} \\ \widetilde{A}\,\mathrm{curl} & 0 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} + iz\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -Ag \\ \widetilde{A}f \end{pmatrix}, \tag{3.10}$$

where for brevity we removed the tilde from the dimensionless fields and displacements. Here z is the dimensionless parameter obtained in (3.7) and (3.8). With A we indicate the inverse of the relative electric permittivity and with $\widetilde{A}$ the inverse of the relative magnetic permeability. Furthermore g is a divergence-free function which represents the external currents of the system, and f is a divergence-free auxiliary function. Note that the non-dimensional constitutive relations are

$$D = A^{-1}E, \quad B = \widetilde{A}^{-1}H.$$

In the same spirit of the work [14] by Birman and Solomyak and the work [8] by Birman and Suslina, we label by $\mathcal{M}$ the Maxwell operator given by the differential expression

$$\begin{pmatrix} 0 & -A\,\mathrm{curl} \\ \widetilde{A}\,\mathrm{curl} & 0 \end{pmatrix}$$

acting on the domain

$$\mathrm{dom}(\mathcal{M}) = \{(E,H):\ \mathrm{div}\,E = 0,\ \mathrm{div}\,H = 0,\ A\,\mathrm{curl}\,H \in L^2(\mathbb{R}^3),\ \widetilde{A}\,\mathrm{curl}\,E \in L^2(\mathbb{R}^3)\}.$$

To have the solvability of (3.10), we need z far from the spectrum of $\mathcal{M}$. A solution pair (E,H) for the Maxwell time-harmonic system exists for $z \in \rho(\mathcal{M})$, where $\rho(\mathcal{M})$ is the resolvent of $\mathcal{M}$. In the present work, we assume that $z \in K \cap \rho(\mathcal{M})$, where K is a compact subset of $\mathbb{R}^3$, in order to prove our estimates. To satisfy this requirement, we set $z = -\mathrm{i}$, however, estimates analogous to those we derive in Chapters 4-6 are valid for $z \in K \cap \rho(\mathcal{M})$ uniformly, i.e. with z-independent constants.

The system of equations we analyse in the present thesis is

$$\begin{cases} A \operatorname{curl} H - E = Ag, \\ \widetilde{A} \operatorname{curl} E + H = \widetilde{A} f. \end{cases} \tag{3.11}$$

To formulate the system of Maxwell equations in the setting of singular periodic structures, we introduce the notion of differentiability of functions that are square integrable with respect to a general Borel measure. In our approach to this task we follow the works [71], [72] and [73] by Zhikov. In Section 3.1 we define the Sobolev spaces with respect to an arbitrary Borel measure and their properties. Furthermore, we introduce two definitions of curl with respect to an arbitrary Borel measure, in order to replace the integration by parts with a formula that works in the setting of arbitrary measures. In the second part of the chapter, we analyse the system of Maxwell equations through two different approaches. In the same spirit of Section 1.2, in Section 3.2 we adapt the method developed by Kamotski and Smyshlyaev in [36], based on the two-scale convergence technique, to the non-magnetic system of Maxwell equations with zero external currents ($\widetilde{A} = I$ and $g = 0$). We prove that the solution of the Maxwell system weakly converges to the solution of the homogenised equation. Section 3.3 and Section 3.4 are devoted to the construction of the homogenised equation for the system of Maxwell equations in the setting of singular periodic structures through the method of multiple-scales asymptotic expansions. This technique takes its origin in the works by Sanchez–Palencia [52] and Bakhvalov [3], [4]. We provide a formal asymptotic expansion and we construct the homogenised equation for the non-magnetic system of Maxwell equations in the case of zero external current ($\widetilde{A} = I$ and $g = 0$) and in the case of non-zero external current ($\widetilde{A} = I$ and $f = 0$), and for the full Maxwell system.

3.1 The set of curls with respect to a measure

The aim of this section is to describe the mathematical framework necessary in order to work with the system (3.11) on singular periodic structures. The main difference with Section 1.1 is the vectorial nature of the electromagnetic fields. Here we describe the property of differentiability of a square integrable vector functions with respect an arbitrary Borel measure. In particular in this chapter we analyse a vectorial problem for the Maxwell system, hence we introduce the notion of μ-curl, that is a curl with respect an arbitrary measure.

Let $Q = [1, 0)^3$ the periodicity cell and let μ an arbitrary Q-periodic Borel measure such that $\mu(Q) = 1$, our first definition is about Sobolev spaces with respect μ.

Definition 3.1.1. *The space $H^1_{\mathrm{curl}} = H^1_{\mathrm{curl}}(Q, d\mu)$ is defined as the closure of the set of* *pairs* $\{(\phi, \operatorname{curl}\phi), \phi \in C^\infty_\#(Q;\mathbb{C}^3)\}$ *in the product* $L^2(Q,d\mu;\mathbb{C}^3) \times L^2(Q,d\mu;\mathbb{C}^3)$, *where* $C^\infty_\#(Q;\mathbb{C}^3) = [C^\infty_\#]^3$ *denotes the set of* Q*-periodic* $C^\infty(\mathbb{R}^3)$ *functions, with values in* $\mathbb{C}^3$.

Elements of this closure are the pairs (u, v), *where* u *and* v *are a vector-valued functions, such that*

$$\exists \phi_n \in C^\infty_\#(Q;\mathbb{C}^3): \qquad \int_Q |\phi_n - u|^2 d\mu \to 0 \qquad \int_Q |\operatorname{curl}\phi_n - v|^2 d\mu \to 0. \tag{3.12}$$

As a particular case of Definition 3.1.1, we say that g is a μ-curl of zero and we write $g \in \mathcal{C}^\mu(0)$, whenever

$$\exists \phi_n \in C^\infty_\#(Q;\mathbb{C}^3): \qquad \int_Q |\phi_n|^2 d\mu \to 0, \qquad \int_Q |g - \operatorname{curl}\phi_n|^2 d\mu \to 0. \tag{3.13}$$

The element v in (3.12) is referred to as a μ-curl of u. We use the notation $\mathcal{C}^\mu(u)$ for the set of μ-curls of u, so we have that $v \in \mathcal{C}^\mu(u)$. When we mention a specific μ-curl, we use the notation $\operatorname{curl}^\mu u$.

Further, note that the set of μ-curl of $u \in H^1_{\operatorname{curl}}(Q,d\mu)$ has the linear structure of the subspace $\mathcal{C}^\mu(0)$ shifted by a $\operatorname{curl}^\mu u$.

Proposition 3.1.1. *For all* $u \in H^1_{\operatorname{curl}}(Q,d\mu)$ *and any* $\operatorname{curl}^\mu u \in \mathcal{C}^\mu(u)$ *we have*

$$\mathcal{C}^\mu(u) = \operatorname{curl}^\mu u + \mathcal{C}^\mu(0) \tag{3.14}$$

where the right hand side denotes the set $\{\operatorname{curl}^\mu u + w \mid w \in \mathcal{C}^\mu(0)\}$.

Proof. The equality (3.14) is understood in sense that for $v \in \mathcal{C}^\mu(u)$, one has $v - \operatorname{curl}^\mu u \in \mathcal{C}^\mu(0)$. Indeed using Definition 3.1.1, there exist $\{\phi_n\}$, $\{\tilde{\phi}_n\} \subset C^\infty_\#(Q;\mathbb{C}^3)$ such that (3.12) holds and

$$\int_Q |u - \tilde{\phi}_n|^2 d\mu \to 0, \qquad \int_Q |\operatorname{curl}^\mu u - \operatorname{curl}\tilde{\phi}_n|^2 d\mu \to 0.$$

For $v - \operatorname{curl}^\mu u$, $\{\phi_n - \tilde{\phi}_n\}$ is an appropriate approximating sequence. Indeed one has $\int_Q |\phi_n - \tilde{\phi}_n|^2 d\mu \to 0$ and

$$\begin{aligned}\int_Q |\operatorname{curl}(\phi_n - \tilde{\phi}_n) - (v - \operatorname{curl}^\mu u)|^2 d\mu &= \int_Q |(\operatorname{curl}\phi_n - v) - (\operatorname{curl}\tilde{\phi}_n - \operatorname{curl}^\mu u)|^2 d\mu \\ &\le \int_Q |\operatorname{curl}\phi_n - v|^2 d\mu + \int_Q |\operatorname{curl}\tilde{\phi}_n - \operatorname{curl}^\mu u|^2 d\mu \to 0.\end{aligned}$$

□

In order to define in a rigorous way the system of Maxwell equations in the setting of singular periodic structures, in analogy with the definition of μ-divergence (see Section 1.1), we introduce an approach which is better suited to the task at hand from the variational perspective.

Definition 3.1.2. *We say that a vector $z \in L^2(Q, d\mu; \mathbb{C}^3)$ and a vector $f \in L^2(Q, d\mu; \mathbb{C}^3)$ are connected by the relation $\widetilde{\operatorname{curl}}^\mu z = f$ if and only if*

$$\int_Q (z \cdot \operatorname{curl} \phi - f \cdot \phi) d\mu = 0 \qquad \forall \phi \in C^\infty_\#(Q; \mathbb{C}^3),$$

where $L^2(Q, d\mu; \mathbb{C}^3)$ is the set of functions square integrable on Q with respect to the measure μ, with values in $\mathbb{C}^3$.

3.1.1 Poincaré-type inequality

An important tool in the study of the system of Maxwell equations, is a Poincaré-type inequality. We define the subspace

$$V_{\text{curl}} = \{v \in H^1_{\text{curl}}(Q, d\mu) : \operatorname{curl}^\mu u = 0\},$$

and we denote with $V_{\text{curl}}^\perp$ the space orthogonal in L^2 sense to V_{curl}. The Poincaré-type inequality states (see [19]) that there exists a constant $C > 0$ such that

$$\|P_{V_{\text{curl}}^\perp} u\|_{L^2(Q,d\mu;\mathbb{C}^3)} \leq C \|\operatorname{curl}^\mu u\|_{L^2(Q,d\mu;\mathbb{C}^3)} \quad \forall u \in H^1_{\text{curl}}(Q, d\mu) \tag{3.15}$$

where $P_{V_{\text{curl}}^\perp}$ is the orthogonal projection onto $V_{\text{curl}}^\perp$ in the L^2 sense.

Note that the bound (3.15) holds in Sobolev spaces with an arbitrary measure μ if we assume, following the idea of Zhikov and Pastukova (see [77, Section 5]) that μ is such that the embedding

$$H^1_{\text{curl}}(Q, d\mu) \subset L^2(Q, d\mu; \mathbb{C}^3)$$

is compact. Indeed, with this assumption we have that the spectrum of the operator $\operatorname{curl} A(\cdot/\varepsilon) \operatorname{curl}$ on the unitary cell Q is discrete and its eigenvalue zero is simple. The constant C in (3.15) is the inverse square root of the lowest eigenvalue of this operator.

3.2 Two-scale convergence analysis in the setting of arbitrary periodic Borel measures

The purpose of this section is to analyse the homogenisation problem for the system of Maxwell equations using the method developed by Kamotski and Smyshyaev in [36]. This approach is originally presented for high contrast PDE system with periodic coefficients in the setting of Lebesgue measure and is based on the two-scale convergence technique. It can be extended in a relatively direct way to the setting of arbitrary Borel measure. In Section 1.2 we adapted it to the scalar elliptic problem, here we address the case of a vectorial problem for the system of Maxwell equations.

The role of this section is to understand with the method developed by Kamotski and Smyshyaev, the structure of the limit term in the two-scale asymptotic, and its relation to the Helmholtz decomposition of square integrable vector functions.

In this section we consider the case of the system of Maxwell equations with relative magnetic permeability set to unity and zero external currents, that is $\widetilde{A} = 1$ and $g = 0$ in the problem (3.11). Let $u^\varepsilon \in H^1_{\text{curl}}(\Omega, d\mu^\varepsilon)$ where $\Omega \subseteq \mathbb{R}^3$, be the solution of

$$\text{curl} A^\varepsilon \,\text{curl}\, u^\varepsilon + u^\varepsilon = f^\varepsilon \in L^2(\Omega, d\mu^\varepsilon; \mathbb{C}^3), \tag{3.16}$$

where $u^\varepsilon = 0$ on $\partial\Omega$. The function f^ε is divergence-free. Note that μ^ε is defined in (1.1), and $A^\varepsilon(\cdot) = A(\frac{\cdot}{\varepsilon})$, where A is a measurable, periodic, bounded, positive definite, symmetric matrix-valued function, satisfying the condition

$$\gamma|\xi|^2 \leq A^\varepsilon(x)\xi \cdot \xi \leq \gamma^{-1}|\xi|^2 \qquad \forall x \in \Omega \quad \gamma > 0,$$

$\varepsilon > 0$ is a small parameter. For a fixed $\varepsilon > 0$ the problem (3.16) is understood as

$$\int_\Omega A^\varepsilon \,\text{curl}\, u^\varepsilon \cdot \text{curl}\, \phi d\mu^\varepsilon + \int_\Omega u^\varepsilon \cdot \phi d\mu^\varepsilon = \int_\Omega f^\varepsilon \cdot \phi d\mu^\varepsilon \qquad \forall \phi \in C_0^\infty(\Omega; \mathbb{C}^3). \tag{3.17}$$

Throughout the section we drop the superscription ε in f^ε for brevity.

3.2.1 A priori estimate and two-scale convergence

In order to obtain the two-scale convergence (see Definition 1.2.1) for u^ε and to construct its limit problem, we start by deriving a priori estimates.

Lemma 3.2.1. *The following a priori estimates hold:*

$$\|u^\varepsilon\|_{L^2(\Omega,d\mu^\varepsilon)} \leq C\|f\|_{L^2(\Omega,d\mu^\varepsilon)}, \tag{3.18}$$

$$\|\,\text{curl}\, u^\varepsilon\|_{L^2(\Omega,d\mu^\varepsilon)} \leq C\|f\|_{L^2(\Omega,d\mu^\varepsilon)}, \tag{3.19}$$

$$\|A^{\frac{1}{2}}\,\text{curl}\, u^\varepsilon\|_{L^2(\Omega,d\mu^\varepsilon)} \leq C\|f\|_{L^2(\Omega,d\mu^\varepsilon)}, \tag{3.20}$$

where with C we indicate a generic positive f-independent constant that can change from line to line.

Proof. Choosing $\phi = u^\varepsilon$ in the formulation (3.17), we have

$$\int_\Omega A^\varepsilon|\,\text{curl}\, u^\varepsilon|^2 d\mu^\varepsilon + \int_\Omega |u^\varepsilon|^2 d\mu^\varepsilon = \int_\Omega f \cdot u^\varepsilon d\mu^\varepsilon.$$

Note that all terms on the left-hand side are non-negative. Furthermore, we can estimate the right-hand side as follows

$$\int_\Omega f \cdot u^\varepsilon d\mu^\varepsilon \leq \frac{1}{2}\gamma \|u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} + \frac{1}{2\gamma}\|f\|^2_{L^2(\Omega, d\mu^\varepsilon)}.$$

So we have

$$\gamma \| \operatorname{curl} u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} + \|u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} \leq \frac{1}{2}\gamma \|u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)} + \frac{1}{2\gamma}\|f\|^2_{L^2(\Omega, d\mu^\varepsilon)},$$

from which (3.18) and (3.19) follow. To obtain (3.20) it suffices to recall that

$$\int_\Omega A \operatorname{curl} u^\varepsilon \cdot \operatorname{curl} u^\varepsilon d\mu^\varepsilon = \|A^{\frac{1}{2}} \operatorname{curl} u^\varepsilon\|^2_{L^2(\Omega, d\mu^\varepsilon)}.$$

□

We introduce now two closed linear subspaces of $L^2(Q, d\mu; \mathbb{C}^3)$:

$$V_{\mathrm{curl}} \doteq \{v \in H^1_{\mathrm{curl}}(Q, d\mu) : \operatorname{curl}^\mu v = 0\}, \tag{3.21}$$

$$W_{\mathrm{curl}} \doteq \{w \in L^2(Q, d\mu; \mathbb{C}^3) : \operatorname{curl}^\mu (A^{\frac{1}{2}} w) = 0\}. \tag{3.22}$$

Lemma 3.2.2. *There exist $u_0(x, y) \in L^2(\Omega; V_{\mathrm{curl}})$ and $\xi_0(x, y) \in L^2(\Omega; W_{\mathrm{curl}})$ such that, up to extracting a subsequence in ε*

$$u^\varepsilon \xrightharpoonup{2} u_0(x, y), \tag{3.23}$$

$$\operatorname{curl} u^\varepsilon \xrightharpoonup{2} \operatorname{curl}_y u_0(x, y), \tag{3.24}$$

$$A^{\frac{1}{2}} \operatorname{curl} u^\varepsilon \xrightharpoonup{2} \xi_0(x, y). \tag{3.25}$$

Proof. The a priori estimates (3.18)-(3.20) imply, up to extracting a subsequence in ε, the existence of the two-scale limits in the measure case as proved in [71, Prop. 2.2]. So there exist $u_0(x, y)$, $\xi_0(x, y) \in L^2(\Omega; Q, d\mu)$ such that satisfy (3.23)-(3.25). We need to show that for x a.e. $u_0(x, y) \in V_{\mathrm{curl}}$ and $\xi_0(x, y) \in W_{\mathrm{curl}}$.

We choose in (3.17) $\phi(x) = \phi^\varepsilon(x) = \varepsilon \Phi(x, \varepsilon^{-1}x)$, $\forall \Phi \in C_0^\infty(\Omega; C^\infty_\#(Q))$, so

$$\int_\Omega A^\varepsilon \operatorname{curl} u^\varepsilon \cdot \varepsilon \operatorname{curl} \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon + \int_\Omega u^\varepsilon \cdot \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon = \int_\Omega f \cdot \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon.$$

In the left hand side we combine (3.25) with the chain rule $\operatorname{curl} \Phi(x, \varepsilon^{-1}x) = \operatorname{curl}_x \Phi +$

$\varepsilon^{-1}\operatorname{curl}_y \Phi$, to obtain that

$$\int_\Omega A^\varepsilon \operatorname{curl} u^\varepsilon \cdot \varepsilon \operatorname{curl} \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to \int_\Omega \int_Q A^{\frac{1}{2}} \xi_0(x,y) \cdot \operatorname{curl}_y \Phi(x,y) d\mu dx.$$

Furthermore

$$\int_\Omega u^\varepsilon \cdot \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to 0 \qquad \forall \Phi \in C_0^\infty(\Omega; C_\#^\infty(Q)).$$

For the right hand side we have

$$\int_\Omega f \cdot \varepsilon \Phi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to 0 \qquad \forall \Phi \in C_0^\infty(\Omega; C_\#^\infty(Q)).$$

So we obtain that $\int_\Omega \int_Q A^{\frac{1}{2}} \xi_0(x,y) \cdot \operatorname{curl}_y \Phi(x,y) d\mu dx = 0$. By Definition 3.1.2 we have

$$\int_\Omega \int_Q A^{\frac{1}{2}} \xi_0(x,y) \cdot \operatorname{curl}_y \Phi(x,y) d\mu dx = \int_\Omega \int_Q \widetilde{\operatorname{curl}}_y A^{\frac{1}{2}} \xi_0(x,y) \cdot \Phi(x,y) d\mu dx = 0,$$

hence $\xi_0(x,y) \in L^2(\Omega; W_{\text{curl}})$. In a similar way we show the regularity of $u_0(x,y)$, in fact using (3.24) we have $\forall \phi \in C_0^\infty(\Omega; C_\#^\infty(Q))$

$$\int_\Omega (A^\varepsilon)^{\frac{1}{2}} \operatorname{curl} u^\varepsilon \cdot \phi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to \int_\Omega \int_Q A^{\frac{1}{2}} \operatorname{curl}_y u_0(x,y) \cdot \phi(x,y) d\mu dx.$$

In other hand, the equation (3.20) ensures that $\|(A^\varepsilon)^{\frac{1}{2}} \varepsilon \operatorname{curl} u^\varepsilon\|_{L^2(\Omega, d\mu^\varepsilon)} \to 0$. This implies that

$$\int_\Omega \int_Q A^{\frac{1}{2}} \operatorname{curl}_y u_0(x,y) \cdot \phi(x,y) d\mu dx = 0 \qquad \forall \phi \in C_0^\infty(\Omega; C_\#^\infty(Q)),$$

hence $A(y)^{\frac{1}{2}} \operatorname{curl}_y u_0(x,y) = 0$ for x a.e., thus we have $u_0(x,y) \in L^2(\Omega, V_{\text{curl}})$. $\square$

3.2.2 Auxiliary theorems for two-scale limits

The next step is to understand the connection between $u_0(x,y)$ and $\xi_0(x,y)$, respectively the limit fields and the limit fluxes. Let indicate with $(\cdot,\cdot)$ the inner product in $H^1_{\text{curl}}(Q, d\mu)$, and with $V_{\text{curl}}^\perp$ the orthogonal complement to V_{curl} defined as

$$V_{\text{curl}}^\perp \doteq \{w \in H^1_{\text{curl}}(Q, d\mu) : \ (w, v) = 0 \ \forall v \in V_{\text{curl}}\}. \tag{3.26}$$

The subspaces V_{curl} and $V_{\text{curl}}^\perp$ are closed, so we can write $H^1_{\text{curl}}(Q, d\mu) = V_{\text{curl}} \oplus V_{\text{curl}}^\perp$.

In order to pass to the limit in equation (3.16), we need to adapt Theorem 1.2.4 and

Theorem 1.2.5 to the curl operator. To do it we need a Poincaré-type inequality such that for $v \in H^1_{\text{curl}}(Q, d\mu)$ holds

$$\|P_{V_{\text{curl}}^\perp} v\|_{L^2(Q,d\mu;\mathbb{C}^3)} \leq C\|\operatorname{curl} v\|_{L^2(Q,d\mu;\mathbb{C}^3)}, \tag{3.27}$$

where $C > 0$ is a constant, and $P_{V_{\text{curl}}^\perp}$ is the orthogonal projection in L^2 sense on $V_{\text{curl}}^\perp$.

Theorem 3.2.3. *The problem on the periodicity cell Q for $v \in H^1_{\text{curl}}(Q, d\mu)$*

$$\operatorname{curl}(A \operatorname{curl} v) = F \in L^2(Q, d\mu)^3,$$

understood as

$$\int_Q A \operatorname{curl} v \cdot \operatorname{curl} w d\mu = \int_Q F \cdot w d\mu \qquad \forall w \in C^\infty_\#(Q), \tag{3.28}$$

is uniquely solvable in $V_{\text{curl}}^\perp$ if and only if $\langle F, w\rangle = 0$ $\forall w \in V_{\text{curl}}$. Furthermore if v is a solution and $v_1 \in V_{\text{curl}}$, then $v + v_1$ is also a solution.

Proof. If v is a solution of (3.28) we choose $w \in V_{\text{curl}}$ so we have

$$\langle F, w\rangle = \int_Q A \operatorname{curl} v \cdot \operatorname{curl} w d\mu = 0.$$

Conversely, let $\langle F, w\rangle = 0$ and look for v such that is a solution of (3.28). This holds if $w \in V_{\text{curl}}$, so we have to prove it for all $w \in V_{\text{curl}}^\perp$. Choosing $V_{\text{curl}}^\perp$ as Hilbert space with the inherited norm $\|\cdot\|_{V_{\text{curl}}^\perp}$ (i.e. the H^1_{curl} norm), we can apply the Lax-Milgram theorem to the bilinear form

$$B(v, w) = \int_Q A \operatorname{curl} v \cdot \operatorname{curl} w d\mu \qquad \forall v, w \in V_{\text{curl}}^\perp.$$

With standard calculation we obtain the continuity, in fact using that $A(y)$ is bounded, we have $|B(v, w)| \leq C\|v\|_{H^1_{\text{curl}}}\|w\|_{H^1_{\text{curl}}}$ $\forall v, w \in V_{\text{curl}}^\perp$, for some $C > 0$. Using the assumption (3.27), the coercivity $B(v, v) \geq C\|v\|^2_{H^1_{\text{curl}}}$ holds $\forall v \in V_{\text{curl}}^\perp$. Hence, there exists a unique solution $v \in V_{\text{curl}}^\perp$ for the problem $\langle F, w\rangle = B(v, w)$ $\forall w \in V_{\text{curl}}^\perp$, that is the (3.28).

To prove that if v is a solution and $v_1 \in V_{\text{curl}}$, then $v + v_1$ is a solution, we can use a classical argument by contradiction. Indeed, assuming u_1 and u_2 solutions of (3.28), set $v = u_1 - u_2$ this solves the problem with $F = 0$. Choosing $w = v$ in (3.28), we have

$$0 = \int_Q A \operatorname{curl} v \cdot \operatorname{curl} v d\mu = \|A^{\frac{1}{2}} \operatorname{curl}_y v\|^2_{L^2(Q,d\mu)},$$

then $v \in V_{\mathrm{curl}}$. $\square$

Theorem 3.2.4. *Under the assumption* (3.27), *let* $v \in L^2(Q, d\mu; \mathbb{C}^3)$ *such that* $v \in W_{\mathrm{curl}}^{\perp}$. *Then exists* $u_1 \in H^1_{\mathrm{curl}}(Q, d\mu)$ *such that* $v = A^{\frac{1}{2}} \operatorname{curl} u_1$. *Where* u_1 *is uniquely defined on* $V_{\mathrm{curl}}^{\perp}$.

Proof. Starting from $v = A^{\frac{1}{2}} \operatorname{curl} u_1$, multiplying by $A^{\frac{1}{2}}$ and applying the curl on both sides, we have

$$\operatorname{curl}(A^{\frac{1}{2}} v) = \operatorname{curl}(A \operatorname{curl} u_1). \tag{3.29}$$

Setting $F \doteq \operatorname{curl}(A^{\frac{1}{2}} v)$, in order to prove the existence of solution u_1, we need to verify that $\forall w \in V_{\mathrm{curl}}$ $\langle F, w \rangle = 0$. By Definition 3.1.2 we have

$$\langle F, w \rangle = \int_Q \operatorname{curl} A^{\frac{1}{2}} v \cdot w d\mu = \int_Q A^{\frac{1}{2}} v \cdot \operatorname{curl} w d\mu,$$

that is zero μ-a.e. since $w \in V_{\mathrm{curl}}$. Hence, there exists a unique $u_1 \in V_{\mathrm{curl}}^{\perp}$ which solves problem (3.29). Now we verify that u_1 satisfies the characterisation of $v \in W_{\mathrm{curl}}^{\perp}$. We have

$$\begin{aligned} \|v - A^{\frac{1}{2}} \operatorname{curl} u_1\|^2_{L^2(Q,d\mu)} &= \langle v, v - A^{\frac{1}{2}} \operatorname{curl} u_1 \rangle - \langle A^{\frac{1}{2}} \operatorname{curl} u_1, v - A^{\frac{1}{2}} \operatorname{curl} u_1 \rangle \\ &\doteq S_1 + S_2, \end{aligned}$$

but from (3.29) it follows that $v - A^{\frac{1}{2}} \operatorname{curl} u_1 \in W_{\mathrm{curl}}$, then $S_1 = 0$ since $v \in W_{\mathrm{curl}}^{\perp}$. On the other hand, relabelling $\phi \doteq v - A^{\frac{1}{2}} \operatorname{curl} u_1$, we have

$$S_2 = \int_Q A^{\frac{1}{2}} \operatorname{curl} u_1 \cdot \phi d\mu = -\int_Q u_1 \cdot \operatorname{curl} A^{\frac{1}{2}} \phi d\mu,$$

that is zero since $\phi \in W_{\mathrm{curl}}$. So we have proved that $\|v - A^{\frac{1}{2}} \operatorname{curl}_y u_1\|^2_{L^2(Q,d\mu)} = 0$, that implies the characterisation of v. $\square$

3.2.3 The limit problem

The link between $u_0(x, y)$ and $\xi_0(x, y)$ is finally explained through the following lemma.

Lemma 3.2.5. *Let* $u_0(x, y)$ *and* $\xi_0(x, y)$ *the two-scale limits defined above. They are such that*

$$\int_\Omega \int_Q \xi_0(x, y) \cdot \Psi(x, y) dx d\mu = \int_\Omega \int_Q u_0(x, y) \cdot \operatorname{curl}_x(A^{\frac{1}{2}} \Psi(x, y)) dx d\mu, \tag{3.30}$$

$\forall \Psi(x, y) \in C^\infty(\Omega; W_{\mathrm{curl}})$.

Proof. Let $\Psi(x,y) = g(x)\psi(y)$ for $g \in C_0^\infty(\Omega)$ and $\psi \in W_{\mathrm{curl}}$. By (3.25) we have

$$\int_\Omega A^{\frac{1}{2}} \operatorname{curl} u^\varepsilon(x) \cdot \Psi(x, \varepsilon^{-1}x) d\mu^\varepsilon \to \int_\Omega \int_Q \xi_0(x,y) \cdot \Psi(x,y) d\mu dx.$$

By Definition 3.1.2 and using the symmetry of A, one has

$$\int_\Omega A^{\frac{1}{2}} \operatorname{curl} u^\varepsilon(x) \cdot \Psi(x, \varepsilon^{-1}x) d\mu^\varepsilon = -\int_\Omega u^\varepsilon(x) \cdot \operatorname{curl}(A^{\frac{1}{2}} \Psi(x, \varepsilon^{-1}x)) d\mu^\varepsilon,$$

and using (3.23)

$$\int_\Omega u^\varepsilon(x) \cdot \operatorname{curl}(A^{\frac{1}{2}} \Psi(x, \varepsilon^{-1}x)) d\mu^\varepsilon \to \int_\Omega \int_Q u_0(x,y) \cdot \operatorname{curl}_x(A^{\frac{1}{2}} \Psi(x,y)) d\mu dx.$$

Comparing the equations we have identity (3.30). □

As consequence of equation (3.30), we define the following linear subspace

$$\begin{aligned} U_{\mathrm{curl}} \doteq \{ & u(x,y) \in L^2(\Omega; V_{\mathrm{curl}}) : \ \exists \xi(x,y) \in L^2(\Omega; W_{\mathrm{curl}}) \text{ such that } \forall \Psi(x,y) \in C^\infty(\Omega; W_{\mathrm{curl}}) \\ & \int_\Omega \int_Q \xi(x,y) \cdot \Psi(x,y) dx d\mu = \int_\Omega \int_Q u(x,y) \cdot \operatorname{curl}_x(A^{\frac{1}{2}} \Psi(x,y)) dx d\mu \}. \end{aligned} \tag{3.31}$$

Therefore we can associate $\xi_0(x,y)$ to any $u_0(x,y)$, so it is possible to define an operator $T : U_{\mathrm{curl}} \to L^2(\Omega; W_{\mathrm{curl}})$ such that $Tu_0(x,y) = \xi_0(x,y)$. An explicit formula for T follows from (3.30), in fact

$$\xi_0(x,y) = Tu_0(x,y) \doteq P_{W_{\mathrm{curl}}}[A^{\frac{1}{2}} \operatorname{curl}_x u_0(x,y)] \in L^2(\Omega; W_{\mathrm{curl}}). \tag{3.32}$$

Note that $C_0^\infty(\Omega; V_{\mathrm{curl}}) \subset U_{\mathrm{curl}}$, hence we have that if $u_0(x,y) \in C_0^\infty(\Omega; V_{\mathrm{curl}})$, $Tu_0(x,y) \in C_0^\infty(\Omega; W_{\mathrm{curl}})$. Further, the following proposition holds.

Proposition 3.2.6. *Let $\phi_0 \in C_0^\infty(\Omega; V_{\mathrm{curl}})$, then exists a unique corrector $\phi_1 \in C_0^\infty(\Omega; V_{\mathrm{curl}}^\perp)$ such that*

$$T\phi_0(x,y) \doteq P_{W_{\mathrm{curl}}}[A^{\frac{1}{2}} \operatorname{curl}_x \phi_0(x,y)] = A^{\frac{1}{2}}[\operatorname{curl}_x \phi_0(x,y) + \operatorname{curl}_y \phi_1(x,y)], \tag{3.33}$$

and ϕ_1 is the unique solution in $H^1_{\mathrm{curl}}(Q, d\mu)$ of

$$\operatorname{curl}_y(A(\operatorname{curl}_x \phi_0 + \operatorname{curl}_y \phi_1)) = 0.$$

Proof. Let $\phi_0 \in C_0^\infty(\Omega; V_{\mathrm{curl}})$, and define

$$\eta(x,y) \doteq T\phi_0(x,y) - A^{\frac{1}{2}} \operatorname{curl}_x \phi_0(x,y) \in C_0^\infty(\Omega; L^2(Y)).$$

But $\eta(x,y) \in C_0^\infty(\Omega; W_{\text{curl}}^\perp)$, indeed using (3.31), (3.32), and by Definition 3.1.2, we have

$$\int_\Omega \int_Q \eta(x,y) \cdot \Psi(x,y) d\mu dx = 0 \qquad \forall \Psi \in C_0^\infty(\Omega; W_{\text{curl}}).$$

So $\eta(x,\cdot) \in W_{\text{curl}}^\perp$, we are in the hypothesis of Theorem 3.2.4, then exists $\phi_1(x,y) \in C_0^\infty(\Omega; V_{\text{curl}}^\perp)$ such that $\eta(x,y) = A^{\frac{1}{2}} \operatorname{curl}_y \phi_1(x,y)$. Comparing this formula with the definition of η we have the claim. □

We are now ready to formulate the two-scale convergence result. Starting from (3.17), we choose $\phi(x) = \phi_\varepsilon(x) = \phi_0(x, \varepsilon^{-1}x) + \varepsilon\phi(x, \varepsilon^{-1}x)$, so we have

$$\int_\Omega A^\varepsilon \operatorname{curl} u^\varepsilon \cdot \operatorname{curl}(\phi_0(x,\varepsilon^{-1}x) + \varepsilon\phi_1(x,\varepsilon^{-1}x)) d\mu^\varepsilon + \int_\Omega u^\varepsilon \cdot (\phi_0(x,\varepsilon^{-1}x)$$
$$+ \varepsilon\phi_1(x,\varepsilon^{-1}x)) d\mu^\varepsilon = \int_\Omega f \cdot (\phi_0(x,\varepsilon^{-1}x) + \varepsilon\phi_1(x,\varepsilon^{-1}x)) d\mu^\varepsilon.$$

For the right hand side, we have the following two-scale limit

$$\int_\Omega f \cdot (\phi_0(x,\varepsilon^{-1}x) + \varepsilon\phi_1(x,\varepsilon^{-1}x)) d\mu^\varepsilon \to \int_\Omega \int_Q f \cdot \phi_0(x,y) d\mu dx.$$

For the left hand side we have

$$\int_\Omega u^\varepsilon \cdot (\phi_0(x,\varepsilon^{-1}x) + \varepsilon\phi_1(x,\varepsilon^{-1}x)) d\mu^\varepsilon \to \int_\Omega \int_Q u_0(x,y) \cdot \phi_0(x,y) d\mu dx,$$

and

$$\begin{aligned} &\int_\Omega A^\varepsilon \operatorname{curl} u^\varepsilon \cdot \operatorname{curl}(\phi_0(x,\varepsilon^{-1}x) + \varepsilon\phi_1(x,\varepsilon^{-1}x)) d\mu^\varepsilon \\ &= \int_\Omega (A^\varepsilon)^{\frac{1}{2}} \operatorname{curl} u^\varepsilon \cdot (A^\varepsilon)^{\frac{1}{2}} (\operatorname{curl}_x \phi_0 + \varepsilon^{-1} \operatorname{curl}_y \phi_0 + \varepsilon \operatorname{curl}_x \phi_1 + \operatorname{curl}_y \phi_1) d\mu^\varepsilon. \end{aligned}$$

Note that $(A^\varepsilon)^{\frac{1}{2}} \operatorname{curl}_y \phi_0 = 0$ since $\phi_0 \in C_0^\infty(\Omega; V_{\text{curl}})$. Using (3.25), we have

$$\int_\Omega A^\varepsilon \operatorname{curl} u^\varepsilon \cdot \operatorname{curl}(\phi_0 + \varepsilon\phi_1) d\mu^\varepsilon \to \int_\Omega \int_Q \xi_0(x,y) \cdot (A^\varepsilon)^{\frac{1}{2}} (\operatorname{curl}_x \phi_0 + \operatorname{curl}_y \phi_1) d\mu dx.$$

Invoking (3.33) and (3.32), we obtain

$$\int_\Omega \int_Q \xi_0(x,y) \cdot (A^\varepsilon)^{\frac{1}{2}} (\operatorname{curl}_x \phi_0 + \operatorname{curl}_y \phi_1) d\mu dx = \int_\Omega \int_Q T u_0(x,y) \cdot T\phi_0(x,y) d\mu dx.$$

So we have that (3.17) two-scale converges to

$$\int_\Omega \int_Q P_{W_{\text{curl}}}(A^{\frac{1}{2}} \operatorname{curl}_x u_0(x,y)) \cdot P_{W_{\text{curl}}}(A^{\frac{1}{2}}(\operatorname{curl}_x \phi_0(x,y))d\mu dx$$
$$+ \int_\Omega \int_Q u_0(x,y) \cdot \phi_0(x,y) d\mu dx = \int_\Omega \int_Q f \cdot \phi_0(x,y) d\mu dx \quad \forall \phi_0 \in C_0^\infty(\Omega; V_{\text{curl}}). \tag{3.34}$$

This is the weak formulation of the limit problem for $u_0(x,y) \in U$.

The result obtained in this section is the two-scale convergence of u^ε, solution of the problem (3.17), to u_0 solution of the limit problem (3.34), in the setting of singular periodic structures described by an arbitrary periodic measure. In a single process we construct the limit equation and we prove the two-scale convergence. In the next section we apply the method of two-scale asymptotic expansion to the non-magnetic system of Maxwell equations, in order to construct an explicit expansion for the solution. We will see that the leading-order term of the expansion for the solution of equation (3.16), has the same structure of u_0, solution of (3.34).

3.3 Asymptotic expansion for the case $\widetilde{A} = 1$

The purpose of this section is to apply the method of two-scale asymptotic expansion to the Maxwell system with unitary relative magnetic permeability, obtained setting $\widetilde{A} = I$ in the problem (3.11). This approach has been used in the classical books [6] and [74] for the Maxwell system, in the setting of Lebesgue measure. In this section we only provide a formal result: we write an explicit expansion for the solution, we construct the homogenised equation but we do not prove any convergence.

This section is a link between the classical approaches in homogenisation and the new method that we present in Chapters 4, 5 and 6, which revises the notion of two-scale asymptotics.

In what follows we start from the system (3.11) with $\widetilde{A} = I$, and we split it as the sum of two cases: the one with zero external currents, that is with $g^\varepsilon = 0$, and the one with non-zero external currents, that is with $f^\varepsilon = 0$.

3.3.1 The formal expansion for the case $g^\varepsilon = 0$

The first case we consider is the one with zero external currents, so with $g = 0$ in the equation (3.11). The Maxwell system we analyse is

$$\begin{cases} \operatorname{curl} v^\varepsilon + u^\varepsilon = f^\varepsilon \\ A(\cdot/\varepsilon) \operatorname{curl} u^\varepsilon = v^\varepsilon, \end{cases} \tag{3.35}$$

where $f^\varepsilon \in L^2(\Omega, d\mu^\varepsilon; \mathbb{C}^3)$ such that $\operatorname{div} f^\varepsilon = 0$. Throughout the section we drop the superscription ε in f^ε for brevity. Note that u^ε is the divergence-free magnetic field (which coincides with the magnetic induction in this setting), and v^ε is the divergence-free electric field. Hence the problem we are considering is (3.16).
The aim is to find an approximation for $u^\varepsilon \in H^1_{\text{curl}}(\Omega, d\mu^\varepsilon)$ that takes into account the rapid oscillations of the equation's coefficients. We write the solution u^ε of (3.16) in the following way:

$$u^\varepsilon(x) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n\left(x, \frac{x}{\varepsilon}\right),$$

(cf. with (1.26)) where $u_n(x, y)$ is Q-periodic in $y = \frac{x}{\varepsilon}$. Like in the case of the asymptotic expansion for the scalar elliptic equation (see Section 1.3), we refer to $x \in \Omega$ as the slow variable, and $y \in Q$ as the fast one.

We plug $u^\varepsilon(x)$ in (3.16) and using the chain rule $\operatorname{curl} = \operatorname{curl}_x + \varepsilon^{-1} \operatorname{curl}_y$ we obtain the following system of equations in powers of ε:

$$\begin{aligned}
&\operatorname{curl} A \operatorname{curl} u^\varepsilon + u^\varepsilon(x) - f(x) \\
&= \varepsilon^{-2} \operatorname{curl}_y A \operatorname{curl}_y u_0(x, y) \\
&+ \varepsilon^{-1}\big(\operatorname{curl}_y A(\operatorname{curl}_y u_1(x, y) + \operatorname{curl}_x u_0(x, y)) + \operatorname{curl}_x A \operatorname{curl}_y u_0(x, y)\big) \\
&+ \varepsilon^{0}\big(\operatorname{curl}_y A \operatorname{curl}_y u_2(x, y) + \operatorname{curl}_x A \operatorname{curl}_y u_1(x, y) \\
&+ \operatorname{curl}_y A \operatorname{curl}_x u_1(x, y) + \operatorname{curl}_x A \operatorname{curl}_x u_0(x, y) + u_0(x, y) - f(x)\big) + o(\varepsilon) = 0,
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
\operatorname{div} u^\varepsilon(x) =& \varepsilon^{-1} \operatorname{div}_y u_0(x, y) + \varepsilon^0\big(\operatorname{div}_x u_0(x, y) + \operatorname{div}_y u_1(x, y)\big) \\
&+ \varepsilon\big(\operatorname{div}_x u_1(x, y) + \operatorname{div}_y u_2(x, y)\big) + \varepsilon^2 \operatorname{div}_x u_2(x, y) + o(\varepsilon^2) = 0.
\end{aligned} \tag{3.37}$$

The two systems hold if and only if the term of every order of smallness in ε is zero.

3.3.2 Construction of the homogenised equation for the case $g^\varepsilon = 0$

We start the construction of the homogenised equation analysing the term of power ε^{-2} in (3.36). In fact, we have

$$\operatorname{curl}_y A \operatorname{curl}_y u_0(x, y) = 0,$$

which is understood as

$$\int_Q A \operatorname{curl}_y u_0(x, y) \cdot \operatorname{curl}_y \phi(x, y) d\mu = 0 \quad \forall \phi \in C^\infty_\#(Q; \mathbb{C}^3).$$

We choose $\phi(x, y) = u_0(x, y)$ and we use the ellipticity of A, then

$$\int_Q A |\operatorname{curl}_y u_0|^2 d\mu \geq \gamma \int_Q |\operatorname{curl}_y u_0|^2 d\mu = 0.$$

Hence $\operatorname{curl}_y u_0(x,y) = 0$. From the first line of (3.37) we have that also $\operatorname{div}_y u_0(x,y) = 0$. This implies that u_0 is constant in y, so $u_0(x,y) = u_0(x)$.

The term of power ε^{-1} in equation (3.36) is

$$\operatorname{curl}_y A \operatorname{curl}_y u_1(x,y) = -\operatorname{curl}_y A \operatorname{curl}_x u_0(x),$$

that is

$$\int_Q A \operatorname{curl}_y u_1(x,y) \cdot \operatorname{curl}_y \phi(x,y) d\mu = -\int_Q A \operatorname{curl}_x u_0(x) \cdot \operatorname{curl}_y \phi(x,y) d\mu \quad \forall \phi \in C^\infty_\#(Q;\mathbb{C}^3). \tag{3.38}$$

Equation (3.38) has a unique solution in $H^1_{\text{curl}}(Q, d\mu)$ if and only if

$$\langle -\operatorname{curl}_y A(y) \operatorname{curl}_x u_0(x), \phi(x,\cdot)\rangle = 0 \qquad \forall \phi(x,\cdot) \in V_{\text{curl}},$$

where V_{curl} is defined in (3.21). This contidion holds using the Definition 3.1.2, in fact we have

$$\int_Q \operatorname{curl}_y A \operatorname{curl}_x u_0(x) \cdot \phi(x,y) d\mu = \int_Q A \operatorname{curl}_x u_0(x) \cdot \operatorname{curl}_y \phi(x,y) d\mu = 0,$$

since $\phi(x,y) \in V_{\text{curl}}$.

The solution $u_1(x,y)$ can be written using the separation of variables in the following way

$$u_1(x,y) = N(y) \operatorname{curl}_x u_0(x) + \tilde{u}_1(x,y), \tag{3.39}$$

where N is a matrix with columns in $H^1_{\text{curl}}(Q, d\mu)$, and $\tilde{u}_1 \in V_{\text{curl}}$.

The equation for N follows from equation (3.38), hence we have the following "cell problem"

$$\begin{cases} \operatorname{curl} A(\operatorname{curl} N + I) = 0, \\ \operatorname{div} N = 0, \end{cases}$$

where $(\operatorname{curl} N)_{ij} = \varepsilon_{ist} N_{tj,s}$ and $(\operatorname{div} N)_i = N_{si,s}$ for $i,j,s,t = 1,2,3$, following the Levi-Civita notation. Such problem is understood as

$$\int_Q A \operatorname{curl} N \cdot \operatorname{curl} \phi d\mu = -\int_Q A \operatorname{curl} \phi d\mu \quad \forall \phi \in C^\infty_\#(Q;\mathbb{C}^3). \tag{3.40}$$

The fact that $\operatorname{div} N = 0$ follows from the term of power ε^0 in the equation (3.37), in fact we have

$$\operatorname{div}_y N(y) \operatorname{curl}_x u_0(x) + \operatorname{div}_y \tilde{u}_1(x,y) = -\operatorname{div}_x u_0(x).$$

Since $\operatorname{div}_x u_0(x) = 0$, we assume $\tilde{u}_1(x,y) \in V_{\text{curl}}$ such that $\tilde{u}_1(x,y) = \nabla_y K(y) \nabla_x u_0(x)$. Here $\nabla_x u_0(x)$ is a matrix with values in $H^1_\#(Q, d\mu)$ defined as $(\nabla u_0)_{ij} = (u_0)_{j,i}$, and $K(y)$ is a matrix with values in $H^1_\#(Q, d\mu)$ such that $\nabla_y K(y)$ is defined as $(\nabla K)_{ijk} =$

$K_{ij,k}$ and such that solves

$$\triangle_y K(y)\nabla_x u_0(x) = 0.$$

In order to prove the existence and uniqueness of a solution N in $H^1_{\text{curl}}(Q, d\mu)$, we apply Lax-Millgram theorem to the equation (3.40). To have the coercivity for this bilinear form, we assume the Poincaré-type inequality

$$\|P_{V_{\text{curl}}^\perp} N\|_{L^2(Q,d\mu)} \le C\|\operatorname{curl} N\|_{L^2(Q,d\mu)}, \quad C > 0. \tag{3.41}$$

We analyse the ε^0 term of equation (3.36). Checking the solvability conditions we obtain

$$\begin{aligned}\langle \operatorname{curl}_x A \operatorname{curl}_y u_1(x,y) + \operatorname{curl}_y A \operatorname{curl}_x u_1(x,y) + \operatorname{curl}_x A \operatorname{curl}_x u_0(x)\\ +u_0(x) - f(x), \phi(x,y)\rangle = 0 \qquad \forall \phi \in V_{\text{curl}},\end{aligned}$$

that is

$$\begin{aligned}\langle \operatorname{curl}_x A \operatorname{curl}_y N(y) \operatorname{curl}_x u_0(x) + \operatorname{curl}_y A \operatorname{curl}_x N(y) \operatorname{curl}_x u_0(x) + \operatorname{curl}_x A \operatorname{curl}_x u_0(x)\\ +u_0(x) - f(x), \phi\rangle + \langle \operatorname{curl}_x A \operatorname{curl}_y \tilde{u}_1 + \operatorname{curl}_y A \operatorname{curl}_x \tilde{u}_1, \phi\rangle = 0 \qquad \forall \phi \in V_{\text{curl}}.\end{aligned} \tag{3.42}$$

Note that $\langle \operatorname{curl}_x A \operatorname{curl}_y \tilde{u}_1, \phi\rangle = 0$, since $\tilde{u}_1(x) \in V_{\text{curl}}$. Furthermore, $\langle \operatorname{curl}_y A \operatorname{curl}_x \tilde{u}_1, \phi\rangle = 0$, which by Definition 3.1.2 means

$$\int_Q A \operatorname{curl}_x \tilde{u}_1(x,y) \cdot \operatorname{curl}_y \phi(x,y) d\mu = 0,$$

since $\phi \in V_{\text{curl}}$. Analysing the remaining part of (3.42) we have that

$$\int_Q A \operatorname{curl}_x (N(y) \operatorname{curl}_x u_0(x)) \cdot \operatorname{curl}_y \phi(x,y) d\mu = 0 \quad \phi \in V_{\text{curl}}.$$

Hence, (3.42) is equivalent to

$$\langle \operatorname{curl}_x A \operatorname{curl}_y (N(y) \operatorname{curl}_x u_0(x)) + \operatorname{curl}_x A \operatorname{curl}_x u_0(x) + u_0(x), \phi\rangle = \langle f, \phi\rangle,$$

that is

$$\int_Q A(\operatorname{curl}_y N(y) + I) \operatorname{curl}_x u_0(x) \cdot \operatorname{curl}_x \phi d\mu + \int_Q u_0(x) \cdot \phi d\mu = \int_Q f \cdot \phi d\mu \quad \forall \phi \in V_{\text{curl}}.$$

Therefore we obtain the homogenised equation

$$\operatorname{curl} A^{\text{hom}} \operatorname{curl} u_0 + u_0 = f, \tag{3.43}$$

where $A^{\text{hom}} = \int_Q A(\operatorname{curl}_y N(y) + I) d\mu$ is the homogenised matrix. This is the version of the homogenised matrix defined in the Introduction and for the equation (1.34), adapted to the case of Maxwell system.

This conclude the formal construction of the homogenised equation for the vectorial problem (3.16). To make this process rigorous, should be proved a convergence between u^ε and u_0, the solution of (3.43). A natural question is whether the two-scale convergence brings to error estimates, however, in the present chapter we do not address the problem of convergence.

In the setting of Lebesgue measure, classical results (see e.g [74], [6] and [52]) only proved weak convergence. In [8, Chapter 7] Birman and Suslina obtained operator-norm resolvent estimates for the solution of the Maxwell system in the form (3.16) in the setting of whole space $\mathbb{R}^3$ with Lebesgue measure.

We provide an operator-norm convergence estimates for the problem (3.16) in the setting of arbitrary Borel measures in Chapter 4. There, we adapt to the vectorial system the method developed in Chapter 2 for the scalar elliptic problem, and we prove a justification for the asymptotic expansion obtained in the present section with the classical method (see Theorem 4.4.2 and Corollary 4.4.3).

3.3.3 The formal expansion for the case $f^\varepsilon = 0$

We consider now the case where external currents g^ε appear and the auxiliary function f^ε is null. The system of Maxwell equations we analyse in this case is

$$\begin{cases} \operatorname{curl} v^\varepsilon + u^\varepsilon = 0, \\ A(\cdot/\varepsilon) \operatorname{curl} u^\varepsilon - v^\varepsilon = A(\cdot/\varepsilon) g^\varepsilon. \end{cases} \tag{3.44}$$

Here u^ε is the divergence-free magnetic field (which coincides with the magnetic induction in this setting) and v^ε is the divergence-free electric field. We consider the solution $u^\varepsilon(x) \in H^1_{\text{curl}}(\Omega, d\mu^\varepsilon)$ with $\Omega \subseteq \mathbb{R}^3$ of the problem

$$\operatorname{curl} A(\cdot/\varepsilon) \operatorname{curl} u^\varepsilon + u^\varepsilon = \operatorname{curl} A(\cdot/\varepsilon) g^\varepsilon, \tag{3.45}$$

with Dirichelet boundary conditions, where $g^\varepsilon(x) \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ is a divergence-free function and the matrix A is defined as above. The problem is understood in the sense of integral identity

$$\int_\Omega A \operatorname{curl} u^\varepsilon \cdot \operatorname{curl} \phi d\mu^\varepsilon + \int_\Omega u^\varepsilon \cdot \phi d\mu^\varepsilon = \int_\Omega A g^\varepsilon \cdot \operatorname{curl} \phi d\mu^\varepsilon \quad \forall \phi \in C_0^\infty(\Omega; \mathbb{C}^3).$$

Throughout the section we drop the superscription ε in g^ε for brevity. The aim is to find an approximation for $u^\varepsilon(x)$ that takes into account the rapid oscillations of the equation's coefficients. We assume that we can write $u^\varepsilon(x)$, solution of (3.45), as

$$u^\varepsilon(x) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n\left(x, \frac{x}{\varepsilon}\right),$$

$u_n(x,y)$ is Q-periodic in $y \in Q$. As before, we treat $x \in \Omega$ as the "slow" variable, and $y = \frac{x}{\varepsilon}$ as the "fast" one.

Plugging $u^\varepsilon(x)$ in (3.45) and using the chain rule $\operatorname{curl} = \operatorname{curl}_x + \varepsilon^{-1} \operatorname{curl}_y$ we obtain

$$\begin{aligned}
&\operatorname{curl} A \operatorname{curl} u^\varepsilon + u^\varepsilon(x) - \operatorname{curl} Ag(x) \\
&= \varepsilon^{-2} \operatorname{curl}_y A \operatorname{curl}_y u_0(x,y) \qquad (3.46)\\
&+ \varepsilon^{-1}\big(\operatorname{curl}_y A curl_y u_1(x,y) + \operatorname{curl}_y A \operatorname{curl}_x u_0(x,y) + \operatorname{curl}_x A \operatorname{curl}_y u_0(x,y) - \operatorname{curl}_y Ag(x)\big)\\
&+ \varepsilon^0\big(\operatorname{curl}_y A \operatorname{curl}_y u_2(x,y) + \operatorname{curl}_x A \operatorname{curl}_y u_1(x,y)\\
&+ \operatorname{curl}_y A \operatorname{curl}_x u_1(x,y) + \operatorname{curl}_x A \operatorname{curl}_x u_0(x,y) + u_0(x,y) - \operatorname{curl}_x Ag(x)\big) + o(\varepsilon) = 0,
\end{aligned}$$

and also

$$\begin{aligned}
\operatorname{div} u^\varepsilon(x) =& \varepsilon^{-1} \operatorname{div}_y u_0(x,y) + \varepsilon^0\big(\operatorname{div}_x u_0(x,y) + \operatorname{div}_y u_1(x,y)\big) \qquad (3.47)\\
&+ \varepsilon\big(\operatorname{div}_x u_1(x,y) + \operatorname{div}_y u_2(x,y)\big) + \varepsilon^2 \operatorname{div}_x u_2(x,y) + o(\varepsilon^2) = 0.
\end{aligned}$$

As in the argument of Section 3.3.1 we use the fact that (3.46) and (3.47) hold if and only if the terms of the same order in ε vanish.

3.3.4 Construction of the homogenised equation for the case $f^\varepsilon = 0$

The construction of the homogenised equation starts from the analysis of the term of power ε^{-2} in (3.46). In fact, we have

$$\operatorname{curl}_y A \operatorname{curl}_y u_0(x,y) = 0,$$

hence $\operatorname{curl}_y u_0(x,y) = 0$. From the term of order ε^{-1} in (3.47) we have that $\operatorname{div}_y u_0(x,y) = 0$, so $u_0(x,y) = u_0(x)$.

The ε^{-1} term of equation (3.46) is

$$\operatorname{curl}_y A \operatorname{curl}_y u_1(x,y) = -\operatorname{curl}_y A \operatorname{curl}_x u_0(x) + \operatorname{curl}_y Ag(x).$$

To check the solvability conditions for this last equation, we need to verify that

$$\langle -\operatorname{curl}_y A(\operatorname{curl}_x u_0(x) + g(x)), \phi(x,y)\rangle = 0 \qquad \forall \phi(x,y) \in V_{\text{curl}},$$

where V_{curl} is defined in (3.21). This request holds using the Definition 3.1.2 since $\phi(x,y) \in V_{\text{curl}}$. We write $u_1(x,y)$ using the separation of variables. Indeed suppose

$u_1(x,y) = N(y)b(x)$ where $N(y)$ is a matrix valued functions in $H^1_{\mathrm{curl}}(Q,d\mu)$ and $b(x)$ a vector in $L^2(Q,d\mu;\mathbb{C}^3)$ we obtain

$$u_1(x,y) = N(y)(\mathrm{curl}_x\, u_0(x) - g(x)) + \tilde{u}_1(x,y), \tag{3.48}$$

where $\tilde{u}_1 \in V_{\mathrm{curl}}$. The equation for $N(y)$ follows from equation for $u_1(x,y)$. In fact, it is easy to check that $N(y)$ solves the 'cell problem' (3.40), and $\tilde{u}_1$ is defined as in (3.39).

Checking the solvability conditions for the equation of power ε^0 in (3.46), we obtain

$$\begin{aligned}&\langle \mathrm{curl}_x A\,\mathrm{curl}_y\, u_1(x,y) + \mathrm{curl}_y A\,\mathrm{curl}_x\, u_1(x,y) + \mathrm{curl}_x A\,\mathrm{curl}_x\, u_0(x)\\ &+ u_0(x) - \mathrm{curl}_x A g(x), \phi(x,y)\rangle = 0, \quad \forall \phi \in V_{\mathrm{curl}}.\end{aligned} \tag{3.49}$$

Using (3.48) we have

$$\begin{aligned}&\langle \mathrm{curl}_x A\,\mathrm{curl}_y\, N(y)(\mathrm{curl}_x\, u_0(x) + g(x)) + \mathrm{curl}_y A\,\mathrm{curl}_x\, N(y)(\mathrm{curl}_x\, u_0(x) + g(x))\\ &+ \mathrm{curl}_x A\,\mathrm{curl}_x\, u_0(x) + u_0(x) - \mathrm{curl}_x A g(x), \phi\rangle\\ &+ \langle \mathrm{curl}_x A\,\mathrm{curl}_y\, \tilde{u}_1 + \mathrm{curl}_y A\,\mathrm{curl}_x\, \tilde{u}_1, \phi\rangle = 0, \quad \forall \phi \in V_{\mathrm{curl}}.\end{aligned}$$

We note that $\langle \mathrm{curl}_x A\,\mathrm{curl}_y\, \tilde{u}_1, \phi\rangle$ is null since $\tilde{u}_1 \in V_{\mathrm{curl}}$. Furthermore we have that $\langle \mathrm{curl}_y A\,\mathrm{curl}_x\, \tilde{u}_1, \phi\rangle$ and $\langle \mathrm{curl}_y A\,\mathrm{curl}_x\, N(y)(\mathrm{curl}_x\, u_0(x)+g(x)), \phi\rangle$ are null since $\phi \in V_{\mathrm{curl}}$. So we have that (3.49) is equivalent to

$$\langle \mathrm{curl}_x A(\mathrm{curl}_y\, N(y) + I)\,\mathrm{curl}_x\, u_0(x) + u_0(x), \phi\rangle = \langle \mathrm{curl}_x A(\mathrm{curl}_y\, N(y) + I) g(x), \phi\rangle,$$

that is

$$\begin{aligned}&\int_Q A(\mathrm{curl}_y\, N(y) + I)\,\mathrm{curl}_x\, u_0(x)\,\mathrm{curl}_x\, \phi d\mu + \int_Q u_0(x)\phi d\mu\\ &= \mathrm{curl}_x \int_Q A(N(y) + I) g(x)\phi d\mu \quad \forall \phi \in V_{\mathrm{curl}}.\end{aligned}$$

Therefore we obtain the homogenised equation

$$\mathrm{curl}_x A^{\mathrm{hom}}\,\mathrm{curl}_x\, u_0(x) + u_0(x) = \mathrm{curl}_x\, A^{\mathrm{hom}} g(x). \tag{3.50}$$

Here the coefficient $A^{\mathrm{hom}} = \int_Q A(y)(\mathrm{curl}_y\, N(y)+I)d\mu$ is the homogenised matrix already introduced in equation (3.43).

In this section we have addressed the formal construction of the homogenised equation for the problem (3.45). This process becomes rigorous once one establishes any convergence between u^ε and u_0. In the present section we do not deal with the issue of convergence.

However, it is interesting to understand whether the two-scale expansion gives reasonable convergence estimates. In this case, the result given by the formal asymptotic expansion is not operator-norm close to the solution of the problem (3.45), even in the setting of Lebesgue measure. Birman and Suslina achieved such result in [10], where they provided operator-norm resolvent estimates for the system of Maxwell equations in the case of constant magnetic permeability and non-zero external current, in the setting of Lebesgue measure. They proved that the limit term is not only the solution of the formal homogenised system as in the formal approach. In fact, their analysis allows the construction of a special corrector which depends on ε and enters in the leading order term of the approximation.

In Chapter 5 we prove an operator-norm estimates for the non-magnetic Maxwell system with non-zero external current, in the setting of arbitrary Borel measure (see Theorem 5.3.1 and Corollary 5.3.6). In our approach as well we infer that the original solution of (3.45) is not operator-norm close to the solution of the homogenised system constructed with the formal method. However, differently from [10], the construction of a corrector is obtained within the task of the derivation of a suitable Poincaré-type inequality.

3.4 Asymptotic expansion for the general case

In this section we analyse the full Maxwell system where the relative magnetic permeability is an arbitrary matrix-valued function. The method of two-scale asymptotic expansion has been applied to this problem by Guenneau in his work [34], in the setting of Lebesgue measure. Here we formally construct the homogenised system in the setting of singular periodic structures, but we do not tackle the issue of convergence.

The general Maxwell system written in terms of fields is:

$$\begin{cases} \operatorname{curl} u^\varepsilon - A^{-1}(\cdot/\varepsilon)v^\varepsilon = g^\varepsilon, \\ \operatorname{curl} v^\varepsilon + \widetilde{A}^{-1}(\cdot/\varepsilon)u^\varepsilon = f^\varepsilon, \end{cases} \tag{3.51}$$

where f^ε and g^ε are two divergence-free functions in $L^2(\Omega, d\mu^\varepsilon; \mathbb{C}^3)$ for $\Omega \subset \mathbb{R}^3$. Here u^ε is the electric field and v^ε is the magnetic field. In order to construct the homogenised system, without loss of generality, we analyse the case with non-zero external currents ($f^\varepsilon = 0$).

Setting $f^\varepsilon = 0$ in (3.51), we obtain

$$\begin{cases} \operatorname{curl} u^\varepsilon - A^{-1}(\cdot/\varepsilon)v^\varepsilon = g^\varepsilon, \\ \operatorname{curl} v^\varepsilon = -\widetilde{A}^{-1}(\cdot/\varepsilon)u^\varepsilon. \end{cases} \tag{3.52}$$

The system (3.52) can be equivalently written in terms of the electric field $v^\varepsilon \in H^1_{\operatorname{curl}}(\Omega, d\mu^\varepsilon)$ as follows:

$$\operatorname{curl} \widetilde{A} \operatorname{curl} v^\varepsilon + A^{-1}v^\varepsilon = -g^\varepsilon, \quad g^\varepsilon \in L^2(\Omega, d\mu^\varepsilon), \quad \operatorname{div} g^\varepsilon = 0. \tag{3.53}$$

Such problem is understood with the integral identity

$$\int_\Omega \widetilde{A}\,\mathrm{curl}\, v^\varepsilon \cdot \mathrm{curl}\,\phi d\mu^\varepsilon + \int_\Omega A^{-1} v^\varepsilon \cdot \phi d\mu^\varepsilon = -\int_\Omega g^\varepsilon \cdot \phi d\mu^\varepsilon, \quad \forall (\phi, \mathrm{curl}\,\phi) \in H^1_{\mathrm{curl}}(\Omega, d\mu^\varepsilon).$$

3.4.1 The formal expansion

In order to find an approximation for the solution pair $(v^\varepsilon, u^\varepsilon)$ of the system (3.52), we assume that we can write the magnetic and the electric fields as follows:

$$u^\varepsilon(x) \sim \sum_{n=0}^{\infty} \varepsilon^n u_n\Big(x, \frac{x}{\varepsilon}\Big), \quad v^\varepsilon(x) \sim \sum_{n=0}^{\infty} \varepsilon^n v_n\Big(x, \frac{x}{\varepsilon}\Big), \tag{3.54}$$

where $u_n(x, y)$, $v_n(x, y)$, $n = 0, 1, 2, \ldots$, are Q periodic in $y = \frac{x}{\varepsilon} \in Q$.

Plugging the expansion (3.54) for v^ε in (3.53), using the chain rule for the curl and for the divergence, we obtain the following recursive systems for the electric field:

$$\begin{aligned}
&\mathrm{curl}\,\widetilde{A}\,\mathrm{curl}\, v^\varepsilon(x) + A^{-1} v^\varepsilon(x)\\
&= \varepsilon^{-2}\,\mathrm{curl}_y\,\widetilde{A}\,\mathrm{curl}_y\, v_0(x, y) \qquad\qquad (3.55)\\
&+ \varepsilon^{-1}\big(\mathrm{curl}_y\,\widetilde{A}\,\mathrm{curl}_y\, v_1(x, y) + \mathrm{curl}_x\,\widetilde{A}\,\mathrm{curl}_y\, v_0(x, y) + \mathrm{curl}_y\,\widetilde{A}\,\mathrm{curl}_x\, v_0(x, y)\big)\\
&+ \varepsilon^0\big(\mathrm{curl}_y\,\widetilde{A}\,\mathrm{curl}_y\, v_2(x, y) + \mathrm{curl}_x\,\widetilde{A}\,\mathrm{curl}_y\, v_1(x, y)\\
&+ \mathrm{curl}_y\,\widetilde{A}\,\mathrm{curl}_x\, v_1(x, y) + \mathrm{curl}_x\,\widetilde{A}\,\mathrm{curl}_x\, v_0(x, y) + A^{-1} v_0(x, y) + g^\varepsilon\big) + o(\varepsilon) = 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{div}\, A^{-1} v^\varepsilon(x) =& \varepsilon^{-1}\,\mathrm{div}_y\, v_0(x, y) + \varepsilon^0(\mathrm{div}_x\, v_0(x, y) + \mathrm{div}_y\, v_1(x, y))\\
&+ \varepsilon(\mathrm{div}_x\, v_1(x, y) + \mathrm{div}_y\, v_2(x, y)) + o(\varepsilon) = 0.
\end{aligned} \tag{3.56}$$

Note that (3.55) and (3.56) hold only if the terms of every order in ε vanish. Furthermore, plugging the expansions (3.54) into (3.52) we obtain the following equalities:

$$\widetilde{A}(\mathrm{curl}_x\, v_0(x, y) + \mathrm{curl}_y\, v_1(x, y)) = -u_0(x, y), \tag{3.57}$$
$$\widetilde{A}(\mathrm{curl}_x\, v_1(x, y) + \mathrm{curl}_y\, v_2(x, y)) = -u_1(x, y). \tag{3.58}$$

Equipped with the formal equations (3.55)-(3.58), we are ready to construct the homogenised system for (3.52).

3.4.2 Construction of the homogenised equation

To start our analysis, we immediately note that the term of power ε^{-2} of (3.55) implies that $\mathrm{curl}_y\, v_0(x, y) = 0$. Furthermore from the first term of (3.56) we know that

$\operatorname{div} A^{-1}v_0(x,y) = 0$. Hence, we infer that

$$v_0(x,y) = (\nabla_y V(y) + I)v_0(x), \tag{3.59}$$

where $V(y) \in H^1_{\#}(Q, d\mu)$ is the vector function solving the following "cell problem":

$$\operatorname{div}_y A^{-1}(\nabla_y V(y) + I) = 0, \quad \int_Q V d\mu = 0. \tag{3.60}$$

Note that $(\nabla_y V)_{ij} = V_{j,i}$, $i,j = 1,2,3$ and $v_0(x)$ is a vector function constant in y.

To characterise $u_0(x,y)$ we start from the equation (3.57). Applying the curl_y on both sides we have that

$$\operatorname{curl}_y \widetilde{A}(\operatorname{curl}_x v_0(x,y) + \operatorname{curl}_y v_1(x,y)) = -\operatorname{curl}_y u_0(x,y).$$

But from the equation of power ε^{-1} of (3.55), and from the fact that $\operatorname{curl}_y v_0(x,y) = 0$, we have that

$$\operatorname{curl}_y \widetilde{A} \operatorname{curl}_y v_1(x,y) + \operatorname{curl}_y \widetilde{A} \operatorname{curl}_x v_0(x,y) = 0,$$

hence $\operatorname{curl}_y u_0(x,y) = 0$. Furthermore applying $\operatorname{div}_y \widetilde{A}^{-1}$ to equation (3.57), one has

$$-\operatorname{div}_y \widetilde{A}^{-1}u_0(x,y) = \operatorname{div}_y(\operatorname{curl}_x v_0(x,y) + \operatorname{curl}_y v_1(x,y)),$$

which is null since $\operatorname{div}_y \operatorname{curl}_x v_0(x,y) = -\operatorname{div}_x \operatorname{curl}_y v_0(x,y) = 0$. So we have that $u_0(x,y)$ is curl_y-free and $\operatorname{div}_y \widetilde{A}^{-1}$-free, and we can write it as

$$u_0(x,y) = (\nabla_y U(y) + I)u_0(x), \tag{3.61}$$

where the vector function $U(y) \in H^1_{\#}(Q, d\mu)$ solves the following "cell problem":

$$\operatorname{div}_y \widetilde{A}^{-1}(\nabla_y U(y) + I) = 0, \quad \int_Q U d\mu = 0. \tag{3.62}$$

To construct the first equation of the homogenised system, we note that plugging (3.59) and (3.61) in equation (3.57) we have

$$-\widetilde{A}^{-1}(\nabla_y U(y) + I)u_0(x) = \operatorname{curl}_x(\nabla_y V(y) + I)v_0(x) + \operatorname{curl}_y v_1(x,y).$$

Testing it with a function $\phi \in V_{\text{curl}}$, we obtain that

$$-\int_Q \widetilde{A}^{-1}(\nabla_y U(y) + I)u_0(x) \cdot \phi d\mu = \int_Q (\nabla_y V(y) + I)v_0(x) \cdot \operatorname{curl}_x \phi d\mu, \quad \forall \phi \in V_{\text{curl}}.$$

Noting that $\operatorname{curl}_x \nabla_y = -\operatorname{curl}_y \nabla_x$, the $\int_Q \nabla_y V(y)v_0(x) \cdot \operatorname{curl}_x \phi$ vanishes, so we obtain that

$$-\int_Q \widetilde{A}(\nabla_y U(y)+I)\, d\mu\, u_0(x) = \operatorname{curl}_x v_0(x), \tag{3.63}$$

where the coefficient $\int_Q \widetilde{A}(\nabla_y U(y)+I)\, d\mu$ is the inverse of the classical homogenised matrix $\widetilde{A}^{\text{hom}}$ already introduced in the equations (3.43) and (3.50) (see [19, Lemma 4.4]).

To construct the second equation of the homogenised system, we consider the solvability condition for the equation of power ε^{-2} of (3.55), so we have

$$\begin{aligned}
&\int_Q g^\varepsilon(x)\cdot \phi d\mu + \int_Q \widetilde{A}(\operatorname{curl}_y v_1(x,y) + \operatorname{curl}_x v_0(x,y))\cdot \operatorname{curl}_x \phi d\mu \\
&+\int_Q A^{-1} v_0(x,y)\cdot \phi d\mu = 0 \quad \forall \phi \in V_{\text{curl}}.
\end{aligned}$$

Using the equation (3.57) and the definitions (3.59) and (3.61), we can rewrite it as

$$\begin{aligned}
&\int_Q (\nabla_y U(y)+I) u_0(x)\cdot \operatorname{curl}_x \phi d\mu - \int_Q A^{-1}(\nabla_y V(y)+I) v_0(x)\cdot \phi d\mu \\
&= \int_Q g^\varepsilon(x)\cdot \phi d\mu \quad \forall \phi \in V_{\text{curl}}.
\end{aligned}$$

Notice that $\operatorname{curl}_x \nabla_y = -\operatorname{curl}_y \nabla_x$, the expression $\int_Q \nabla_y U(y) u_0(x)\cdot \operatorname{curl}_x \phi$ vanishes since $\phi \in V_{\text{curl}}$, hance we obtain the following equation

$$\operatorname{curl}_x u_0(x) - \int_Q A^{-1}(\nabla_y V(y)+I)\, d\mu\, v_0(x) = g^\varepsilon, \tag{3.64}$$

where the coefficient $\int_Q A^{-1}(\nabla_y V(y)+I)\, d\mu$ is the inverse of the classical homogenised matrix A^{hom} defined in the equations (3.43) and (3.50) (see [19, Lemma 4.4]). The equations (3.63) and (3.64) together form the homogenised system for (3.52):

$$\begin{cases} \operatorname{curl} u_0 - \left(A^{\text{hom}}\right)^{-1} v_0 = g^\varepsilon, \\ \operatorname{curl} v_0 = \left(\widetilde{A}^{\text{hom}}\right)^{-1} u_0. \end{cases} \tag{3.65}$$

In this section we have carried out the formal construction of the homogenised equations for the general Maxwell system. This process, however, cannot be substantiated rigorously. In fact, the traditional approach of two-scale asymptotic expansion gives and incorrect answer for the full Maxwell system, even in the case of the Lebesgue measure.

This result was obtained by Suslina in the works [58] and [60], where she presented operator-norm resolvent estimates for the general Maxwell system (3.51) in the Lebesgue measure setting. With her approach, Suslina constructed a special corrector depending on ε, that enters the leading order term of the approximation.

In Chapter 6 we provide operator-norm resolvent estimates for the full Maxwell system (3.51) in the setting of arbitrary Borel measures. Such result is based on the method developed in Chapter 5 for the non-magnetic Maxwell system with non-zero external currents. We show (see Theorem 6.2.3 and Corollary 6.2.5) that the homogenised problem has a special structure depending both on ε and $y \in Q$, different to the one provided by the formal argument.

In the present chapter, we approached the Maxwell system with a classical technique, as anyone else would approach it, and we carried out the formal construction of the homogenised system for different cases, in Sections 3.3.2, 3.3.4 and 3.4.2. The natural question is is whether this classical method brings to norm-resolvent convergence. This motivate us to think about new ways to representing the multiscale asymptotics in a different perspective. Our method, introduced in Chapter 2 for the scalar elliptic equation (1.7), allows us to provide a justification for the asymptotic result obtained with the formal construction, in Theorem 2.2.3 and in Corollary 2.2.4. This tempts us to try to prove the same kind of estimates for the system of Maxwell equations in the three different cases described by (3.35), (3.44) and (3.51).

However, it turns out to be the right idea only for the case of the non-magnetic Maxwell system with zero external currents (i.e. (3.35)). In fact, as was mentioned at the end of Section 3.3.2, in Chapter 4 we achieve operator-norm estimate for the difference between the solution of (3.35) and the solution of the homogenised problem obtained with the standard formal two-scale asymptotic.

The situation is different for the case of the non-magnetic Maxwell system with non-zero external currents (3.44), and for the full Maxwell system (3.51). In fact, as was mentioned above and at the end of Section 3.3.4, the solutions of the suggested formal homogenised equations are not operator-norm close to the solutions of the original problems. In Chapters 5 and 6 we modify our method and we prove, (see Corollary 5.3.6 and Corollary 6.2.5), that the correct replacement of the standard formal limit system, involves an ε-dependent pseudo-differential operator (see (5.34) and (6.16)), which is in some sense, a singular perturbation of the formal homogenised equation suggested by the two-scale asymptotic approach.

Chapter 4

Operator-norm homogenisation estimates for the non-magnetic system of Maxwell equations: the case of zero external current

Introduction

In the present chapter we prove operator-norm resolvent estimates for the system of Maxwell equations with rapidly oscillating coefficients. This result is contained in the work [20] by Cherednichenko and D'Onofrio. The operator-theoretic perspective on partial differential equations with multiple scales has proved effective for obtaining sharp convergence results for problems of periodic homogenisation, see *e.g.* [70], [32], [8], [26], [65] for related developments in the "whole-space" setting, *i.e.* when the spatial domain is invariant with respect to shifts by the elements of a periodic lattice in $\mathbb{R}^d$, $d \geq 2$. The techniques developed in the above works have highlighted a variety of different new ways to interpret the process homogenisation, *e.g.* via the singular-value decomposition of operator resolvents or by extending the classical perturbation series to PDE families dependent on an additional length-scale parameter. However, a common strand in all of them is the idea that homogenisation corresponds is a "long-wave" asymptotic regime, governed by the behaviour of the related differential operators near the bottom of its spectrum. It seems natural to enquire whether this rationale can be extended to arbitrary periodic (Borel) measures, providing useful order-sharp approximations for periodic "structures".

In Chapter 2 (see [22]) we addressed the above question for the case of a scalar elliptic problem

$$-\nabla\cdot A(\cdot/\varepsilon)\nabla u+u=f,\qquad f\in L^2(\mathbb{R}^d,d\mu^\varepsilon),\qquad \varepsilon>0,$$

where the ε-periodic measure μ^ε is obtained by scaling from a given 1-periodic measure μ^1, and the matrix-function A is uniformly positive definite. As a starting point of our approach, we considered the PDE family obtained from (4) by the Floquet transform (see [26], [77]), in some sense replacing the macroscopic variable by an additional parameter θ ("quasimomentum"), akin to the Fourier dual variable for PDE with constant coefficients. The strategy for the analysis of the family obtained was to use an asymptotic approximation for the solution in powers of ε, carefully analyse the homogenisation corrector as a function of ε and θ, and obtain an estimate for the remainder that is uniform with respect to θ. The key technical tool for the proof of remainder estimates was a Poincaré-type inequality in an appropriate Sobolev space of quasiperiodic functions, conditioned by the fact that we deal with an arbitrary measure. Equipped with this new machinery, in the present chapter we set out to tackle a vector problem, in particular the system of Maxwell equations, which is of interest in applications to electromagnetism.

Consider a Q-periodic Borel measure μ in $\mathbb{R}^3$, where $Q=[0,1)^3$, such that $\mu(Q)=1$. For each $\varepsilon>0$ we define the "ε-scaling" of μ, *i.e.* the ε-periodic measure μ^ε given by $\mu^\varepsilon(B)=\varepsilon^3\mu(\varepsilon^{-1}B)$ for all Borel sets $B\subset\mathbb{R}^3$, so that $\mu^1\equiv\mu$. Henceforth, we denote by $L^2(\mathbb{R}^3,d\mu^\varepsilon)$ the space of vector functions with values in $\mathbb{C}^3$ that are square integrable in $\mathbb{R}^3$ with respect to the measure μ^ε.

We aim at analysing the long-scale properties of periodic structures described by the measures μ^ε, in the context of the system of equations of electromagnetism. More precisely, in what follows we analyse the asymptotic behaviour, as $\varepsilon\to 0$, of the solutions u^ε to the vector problems

$$\operatorname{curl}\bigl(A(\cdot/\varepsilon)\operatorname{curl}u^\varepsilon\bigr)+u^\varepsilon=f^\varepsilon,\qquad f^\varepsilon\in L^2(\mathbb{R}^3,d\mu^\varepsilon),\tag{4.1}$$

where A is a real-valued μ-measurable matrix function, assumed to be Q-periodic, symmetric, bounded and uniformly positive definite. The right-hand sides f^ε are assumed to be divergence-free, in the sense that

$$\int_{\mathbb{R}^3}f^\varepsilon\cdot\nabla\phi\,d\mu^\varepsilon=0\qquad\forall\phi\in C_0^\infty(\mathbb{R}^3).\tag{4.2}$$

Henceforth, all function spaces are defined over the field $\mathbb{C}$ of complex numbers.

Equation (4.1) is the resolvent form of the Maxwell system of equations of electromagnetism in the absence of external currents, see [35], [17], where u^ε represents the divergence-free magnetic field H^ε, the matrix A is the inverse of the relative dielectric permittivity, and the relative magnetic permeability is set to unity. The right-hand sides f^ε play an auxiliary rôle: they do not appear in the original Maxwell system but are introduced in this work for purposes of the resolvent analysis of the "reduced" Maxwell operator on the left-hand side of (4.1).

Our goal is to derive order-sharp operator-norm estimates for the difference between u^ε and the solution $u^\varepsilon_{\rm hom}$ of the homogenised equation

$$\operatorname{curl}(A^{\rm hom}\operatorname{curl}u^\varepsilon_{\rm hom})+u^\varepsilon_{\rm hom}=f^\varepsilon,\qquad f^\varepsilon\in L^2(\mathbb{R}^3,d\mu^\varepsilon),\qquad \operatorname{div}f^\varepsilon=0,\tag{4.3}$$

where $A^{\rm hom}$ is a constant matrix representing the effective, or "homogenised", properties of the medium. In other words, we aim at finding a matrix $A^{\rm hom}$ for which there exists $C>0$, independent of ε and f^ε, such that

$$\big\|u^\varepsilon-u^\varepsilon_{\rm hom}\big\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}\le C\varepsilon\|f^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}\qquad\forall\varepsilon\in(0,1].\tag{4.4}$$

Clearly, a matrix $A^{\rm hom}$ with this property is unique. A similar result is obtained in [8, Chapter 7.3] for the case when μ is the Lebesgue measure, using perturbation analysis of the operators in (4.1) near the bottom of the spectrum. Our approach here is based on the asymptotic expansions for solutions to weak formulations, rather than the analysis of spectral properties.

Denote by $C_0^\infty(\mathbb{R}^3)$ the set of infinitely smooth complex-valued vector functions with compact support in $\mathbb{R}^3$. The solutions of (4.1) are understood as pairs $(u^\varepsilon,\operatorname{curl}u^\varepsilon)$ in the space $H^1_{\rm curl}(\mathbb{R}^3,d\mu^\varepsilon)$ defined as the closure of the set of pairs

$$\big\{(\phi,\operatorname{curl}\phi),\ \phi\in C_0^\infty(\mathbb{R}^3)\big\}$$

in the direct sum $L^2(\mathbb{R}^3,d\mu^\varepsilon)\oplus L^2(\mathbb{R}^3,d\mu^\varepsilon)$. We say that $(u^\varepsilon,\operatorname{curl}u^\varepsilon)$ is a solution to (4.1) if

$$\int_{\mathbb{R}^3}A(\cdot/\varepsilon)\operatorname{curl}u^\varepsilon\cdot\overline{\operatorname{curl}\phi}\,d\mu^\varepsilon+\int_{\mathbb{R}^3}u^\varepsilon\cdot\overline{\phi}\,d\mu^\varepsilon=\int_{\mathbb{R}^3}f^\varepsilon\cdot\overline{\phi}\,d\mu^\varepsilon\tag{4.5}$$
$$\forall(\phi,\operatorname{curl}\phi)\in H^1_{\rm curl}(\mathbb{R}^3,d\mu^\varepsilon).$$

Note that for each $\varepsilon>0$ the left-hand side of (4.5) defines an equivalent inner product on $H^1_{\rm curl}(\mathbb{R}^3,d\mu^\varepsilon)$. The right-hand side is a linear bounded functional on $H^1_{\rm curl}(\mathbb{R}^3,d\mu^\varepsilon)$. The existence and uniqueness of u^ε satisfying the integral identity (4.5) is a consequence of the classical Riesz representation theorem for linear functional in a Hilbert space.

In what follows we study the resolvent of the operator $\mathcal{A}^\varepsilon$ with domain

$$\operatorname{dom}(\mathcal{A}^\varepsilon)=\bigg\{u\in L^2(\mathbb{R}^3,d\mu^\varepsilon):\ \exists\operatorname{curl}u\in L^2(\mathbb{R}^3,d\mu^\varepsilon)\text{ such that}$$
$$\int_{\mathbb{R}^3}A(\cdot/\varepsilon)\operatorname{curl}u\cdot\overline{\operatorname{curl}\phi}\,d\mu^\varepsilon+\int_{\mathbb{R}^3}u\cdot\overline{\phi}\,d\mu^\varepsilon=\int_{\mathbb{R}^3}f\cdot\overline{\phi}\,d\mu^\varepsilon\tag{4.6}$$
$$\forall(\phi,\operatorname{curl}\phi)\in H^1_{\rm curl}(\mathbb{R}^3,d\mu^\varepsilon)\quad\text{for some } f\in L^2(\mathbb{R}^3,d\mu^\varepsilon),\ \operatorname{div}f=0\bigg\},$$

defined by the formula $\mathcal{A}^\varepsilon u = f - u$, where $f \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, $\operatorname{div} f = 0$, and $u \in \operatorname{dom}(\mathcal{A}^\varepsilon)$ are linked [1] as in (4.6). Notice that, in general, for a given $u \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ there may be more than one element $(u, \operatorname{curl} u) \in H^1_{\text{curl}}(\mathbb{R}^3, d\mu^\varepsilon)$. However, for each $u \in \operatorname{dom}(\mathcal{A}^\varepsilon)$ there exists exactly one $\operatorname{curl} u$ such that (4.6) holds, which is a consequence of the uniqueness of solution to the integral identity (4.5).

Clearly, the operator $\mathcal{A}^\varepsilon$ is symmetric. Furthermore, similarly to Chapter 2 we infer that $\operatorname{dom}(\mathcal{A}^\varepsilon)$ is dense in $\{u \in L^2(\mathbb{R}^3, d\mu^\varepsilon) : \operatorname{div} u = 0\}$. Indeed, by the definition of $\operatorname{dom}(\mathcal{A}^\varepsilon)$, if $f \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, $\operatorname{div} f = 0$ and $u, v \in \operatorname{dom}(\mathcal{A}^\varepsilon)$ are such that $\mathcal{A}^\varepsilon u + u = f$ and $\mathcal{A}^\varepsilon v + v = u$, one has

$$\int_{\mathbb{R}^3} |u|^2 d\mu^\varepsilon = \int_{\mathbb{R}^3} f\overline{v}\, d\mu^\varepsilon.$$

This identity entails that if f is orthogonal to $\operatorname{dom}(\mathcal{A}^\varepsilon)$, then $u = 0$ and so $f = 0$. It follows from the definition of $\mathcal{A}^\varepsilon$ that its defect numbers are zero, hence it is self-adjoint. Analogously, we define the operator $\mathcal{A}^{\text{hom}}$ associated with the problem (4.3), so that (4.3) holds if and only if $u^\varepsilon_{\text{hom}} = (\mathcal{A}^{\text{hom}} + I)^{-1} f^\varepsilon$.

All integrals and differential operators below, unless indicated otherwise, are understood appropriately with respect to the measure μ. Throughout the chapter we use the notation e_κ for the exponent $\exp(i\kappa \cdot y)$, $y \in Q$, $\varkappa \in [-\pi, \pi)^3$, and a similar notation e_θ for the exponent $\exp(i\theta \cdot x)$, $x \in \mathbb{R}^3$, $\theta \in \varepsilon^{-1}[-\pi, \pi)^3$. We denote by $C^\infty_\#$ the set of Q-periodic functions in $C^\infty(\mathbb{R}^3)$, and $\operatorname{curl} \phi$, $\operatorname{curl}(e_\kappa \phi)$ $\operatorname{curl}(e_{\varepsilon\theta}\phi)$ are the classical curls of smooth functions ϕ, $e_\kappa \phi$, $e_{\varepsilon\theta}\phi$.

4.1 Sobolev spaces of quasiperiodic functions

In this section we recall the definition of the space of quasiperiodic functions with respect an arbitrary Borel measure μ.

Definition 4.1.1. *For each $\kappa \in [-\pi, \pi)^3 =: Q'$, the space $H^1_{\text{curl},\kappa}(Q, d\mu)$ is defined as the closure of the set of pairs $\{(e_\kappa \phi, \operatorname{curl}(e_\kappa \phi)) : \phi \in [C^\infty_\#]^3\}$ with respect the standard norm in $L^2(Q, d\mu) \oplus L^2(Q, d\mu)$. For $(u, v) \in H^1_{\text{curl},\kappa}$ we denote by $\operatorname{curl} u$ the second element v in the pair. We will use the notation $H^1_{\text{curl}}(Q, d\mu) = H^1_{\text{curl}}$ for the space $H^1_{\text{curl},\kappa}$ with $\kappa = 0$.*

Note that there may be different elements in $H^1_{\text{curl},\kappa}$ with the same first component. Indeed for any pair $(u, v) \in H^1_{\text{curl},\kappa}$ and a vector function w obtained as the limit in $L^2(Q, d\mu)$ of $\operatorname{curl}(e_\kappa \phi_n)$ for a sequence $\phi_n \in C^\infty_\#$ converging to zero in $L^2(Q, d\mu)$, the element $(u, v + w)$ is also in $H^1_{\text{curl},\kappa}$. In addition, $H^1_{\text{curl},\kappa}$ and $H^1_{\text{curl},0}$ are related by a one-to-one map. Indeed, for any element $(u, v) \in H^1_{\text{curl},\kappa}$ the couple $(\overline{e_\kappa} u,\ \overline{e_\kappa}(v - i\kappa \times u))$

[1]It is not difficult to show that for each $u \in \operatorname{dom}(\mathcal{A}^\varepsilon)$ there exists only one f with the property described in (4.6).

is in $H^1_{\rm curl}$, which follows from

$$\operatorname{curl}\phi_n = \operatorname{curl}(\overline{e_\kappa} e_\kappa \phi_n) = \overline{e_\kappa}\operatorname{curl}(e_\kappa\phi_n) - {\rm i}\kappa\times\phi_n,$$

for all sequences $\{\phi_n\}$ such that $e_\kappa\phi_n \to e_\kappa u$, $\operatorname{curl}(e_\kappa\phi_n)\to\operatorname{curl}(e_\kappa u)$. Conversely, for all $(\widetilde{u},\widetilde{v})\in H^1_{\rm curl}$ one has $\widetilde{v} = \overline{e_\kappa}(v - i\kappa\times u)$ for some $(u,v)\in H^1_{{\rm curl},\kappa}$.

We say that $F\in L^2(Q,d\mu)$ is divergence-free, or solenoidal, and write $\overline{e_\kappa}\operatorname{div}(e_\kappa F) = 0$, if

$$\int_Q e_\kappa F\cdot\overline{\nabla(e_\kappa\phi)}\,d\mu = 0 \qquad \forall\phi\in C^\infty_\#. \tag{4.7}$$

Now suppose that A is a μ-measurable bounded, symmetric, pointwise positive real-valued matrix function such that A^{-1} is μ-essentially bounded. For each $\kappa\in Q'$ we analyse the operator $\mathcal{A}_\kappa$ with domain

$$\begin{aligned}\operatorname{dom}(\mathcal{A}_\kappa) = \Big\{ & u\in L^2(Q,d\mu):\ \exists\operatorname{curl}(e_\kappa u)\in L^2(Q,d\mu) \text{ such that} \\ & \int_Q A\operatorname{curl}(e_\kappa u)\cdot\overline{\operatorname{curl}(e_\kappa\phi)}\,d\mu + \int_Q u\cdot\overline{\phi}\,d\mu = \int_Q F\cdot\overline{\phi}\,d\mu \quad \forall\phi\in\big[C^\infty_\#\big]^3, \\ & \text{for some } F\in L^2(Q,d\mu),\ \ \overline{e_\kappa}\operatorname{div}(e_\kappa F) = 0\Big\},\end{aligned} \tag{4.8}$$

defined by the formula $\mathcal{A}_\kappa u = F - u$ where $F\in L^2(Q,d\mu)$ and $u\in\operatorname{dom}(\mathcal{A}_\kappa)$ are linked as in (4.8). By an argument similar to the case of $\mathcal{A}^\varepsilon$, the domain $\operatorname{dom}(\mathcal{A}_\kappa)$ is dense in $\{u\in L^2(Q,d\mu) : \overline{e_\kappa}\operatorname{div}(e_\kappa u) = 0\}$, and $\mathcal{A}_\kappa$ is a self-adjoint operator.

We assume throughout the chapter that the measure μ is such that if for $u\in H^1_{\rm curl}(\mathbb{R}^3,d\mu)$ we have $\operatorname{curl}u = 0$, then $u = \nabla\psi + a$ for some $\psi\in H^1(\mathbb{R}^3,d\mu)$ and $a\in\mathbb{C}^3$. Here the space $H^1(\mathbb{R}^3,d\mu)$ is defined, similarly to the space $H^1_{\rm curl}(Q,d\mu)$, as the closure of the set of pairs $\{(\phi,\nabla\phi):\phi\in C^\infty_\#\}$ with respect the standard norm in $L^2(Q,d\mu)\oplus[L^2(Q,d\mu)]^3$. Note that the class of measures μ for which the above condition holds includes [29, p. 219] the Lebesgue measure in $\mathbb{R}^3$ as well as its restriction to the complement of any Q-periodic set ("perforations") with smooth boundary such that the distance for an individual connected component to the boundary of Q is positive.[2]

4.1.1 Floquet transform

In this section we define, as for the scalar case in Chapter 2, a representation for functions in $L^2(\mathbb{R}^3,d\mu^\varepsilon)$ introduced in [31]. In the paper [77], properties of the Gelfand transform with respect to the arbitrary periodic Borel measure μ have been studied and their applications to spectral analysis of elliptic PDEs have been discussed. Here we describe its "Floquet" version, which we then use for the asymptotic analysis of our main equation

[2]In [71] such periodic sets are referred to as "disperse".

(4.1) as $\varepsilon \to 0$. The transform we use is unitary equivalent to Gelfand transform, where the unitary transformation is simply by a multiplication by the function e_θ, whose L^2 norm is unity.

Definition 4.1.2. *For $\varepsilon > 0$ and $u \in C_0^\infty(\mathbb{R}^3)$, the ε-Floquet transform $\mathcal{F}_\varepsilon u$ is the function*

$$(\mathcal{F}_\varepsilon u)(z, \theta) = \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \sum_{n \in Z^d} u(z + \varepsilon n) \exp(-\mathrm{i}\varepsilon n \cdot \theta), \quad z \in \varepsilon Q, \ \theta \in \varepsilon^{-1} Q'.$$

The mapping $\mathcal{F}_\varepsilon$ preserves the norm and can be extended to an isometry

$$\mathcal{F}_\varepsilon : L^2(\mathbb{R}^3, d\mu^\varepsilon) \longrightarrow L^2(\varepsilon^{-1}Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon),$$

which we also refer to as ε-Floquet transform. By following an argument similar to that given in Section 2.1.1, the mapping $\mathcal{F}_\varepsilon$ is shown to be unitary for all $\varepsilon > 0$ and the inverse of $\mathcal{F}_\varepsilon$ is given by

$$(\mathcal{F}_\varepsilon g)^{-1}(z) = \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \int_{\varepsilon^{-1}Q'} g(\theta, z)\, d\theta, \quad z \in \mathbb{R}^d, \qquad g \in L^2(\varepsilon^{-1}Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon).$$

To obtain the version of the Floquet transform that we use in what follows, we combine the ε-Floquet transform with the unitary scaling transform $\mathcal{T}_\varepsilon$ defined by

$$\mathcal{T}_\varepsilon h(\theta, y) := \varepsilon^{3/2} h(\theta, \varepsilon y), \qquad \theta \in \varepsilon^{-1}Q', \ y \in Q, \qquad \forall h \in L^2(\varepsilon^{-1}Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon),$$

$$(\mathcal{T}_\varepsilon^{-1} h)(\theta, z) = \varepsilon^{-3/2} h(\theta, z/\varepsilon), \quad \theta \in \varepsilon^{-1}Q', \ z \in \varepsilon Q, \qquad \forall h \in L^2(\varepsilon^{-1}Q' \times Q, d\theta \times d\mu).$$

Proposition 4.1.3. *For each $\varepsilon > 0$ we have the following unitary equivalence between the operator $\mathcal{A}^\varepsilon$ and the direct integral of the family $\mathcal{A}_{\varepsilon\theta}$, $\theta \in \varepsilon^{-1}Q'$:*

$$(\mathcal{A}^\varepsilon + I)^{-1} = \mathcal{F}_\varepsilon^{-1} \mathcal{T}_\varepsilon^{-1} \left(\int_{\varepsilon^{-1}Q'}^{\oplus} e_{\varepsilon\theta} (\varepsilon^{-2}\mathcal{A}_{\varepsilon\theta} + I)^{-1} \overline{e_{\varepsilon\theta}}\, d\theta \right) \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon,$$

where $\varepsilon\theta = \kappa$.

Sketch of the proof. The argument is similar to that given in [26] and in Chapter 2 for the scalar case. We consider the solution $(u^\varepsilon, \operatorname{curl} u^\varepsilon) \in H^1_{\text{curl}}$ of the problem (4.1) with $f \in C_0^\infty(\mathbb{R}^3)$. We then introduce the "periodic amplitude" of its Floquet transform

$$u_\theta^\varepsilon(y) := \overline{e_{\varepsilon\theta}} \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon u(y) = \left(\frac{\varepsilon^2}{2\pi}\right)^{3/2} \sum_{n \in Z^d} u^\varepsilon(\varepsilon y + \varepsilon n) \exp\bigl(-\mathrm{i}(\varepsilon y + \varepsilon n) \cdot \theta\bigr), \quad y \in Q. \quad (4.9)$$

By approximating u_θ^ε with smooth functions, it is straightforward to see that if, for each choice of $\operatorname{curl} u^\varepsilon$, we write

$$\operatorname{curl}(e_{\varepsilon\theta} u_\theta^\varepsilon)(y) = \varepsilon \left(\frac{\varepsilon^2}{2\pi}\right)^{3/2} \sum_{n \in Z^d} \operatorname{curl} u^\varepsilon(\varepsilon y + \varepsilon n) \exp\bigl(-\mathrm{i}\varepsilon n \cdot \theta\bigr), \qquad y \in Q,$$

then $(e_\kappa u_\theta^\varepsilon, \operatorname{curl}(e_\kappa u_\theta^\varepsilon)) \in H^1_{\operatorname{curl},\kappa}(Q, d\mu)$. Furthermore,

$$\varepsilon^{-2} \int_Q A \operatorname{curl}(e_{\varepsilon\theta} u_\theta^\varepsilon) \cdot \overline{\operatorname{curl}(e_{\varepsilon\theta}\phi)}\, d\mu + \int_Q e_{\varepsilon\theta} u_\theta^\varepsilon \cdot \overline{e_{\varepsilon\theta}\phi}\, d\mu = \int_Q e_{\varepsilon\theta} F \cdot \overline{e_{\varepsilon\theta}\phi}\, d\mu \quad \forall \phi \in \bigl[C_\#^\infty\bigr]^3, \tag{4.10}$$

where $F := \overline{e_{\varepsilon\theta}} \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon f$. It is verified directly that F is solenoidal, *cf.* (4.7). By the density of $f \in C_0^\infty(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3, d\mu^\varepsilon)$, we obtain the claim. $\square$

In what follows, we study the asymptotic behaviour, as $\varepsilon \to 0$, of the solution u_θ^ε to the problem

$$\varepsilon^{-2} \overline{e_{\varepsilon\theta}} \operatorname{curl}\bigl(A \operatorname{curl}(e_{\varepsilon\theta} u_\theta^\varepsilon)\bigr) + u_\theta^\varepsilon = F \qquad \varepsilon > 0, \quad \theta \in \varepsilon^{-1} Q', \tag{4.11}$$

for all solenoidal $F \in L^2(Q, d\mu)$. The problem (4.11) is understood in the sense of the identity (4.10). We will show that u_θ^ε is ε-close with respect to the norm of $L^2(Q, d\mu)$, uniformly in $\theta \in \varepsilon^{-1}Q'$ to the constant vector c_θ solving the "homogenised" equation related to (4.11):

$$\theta \times A^{\mathrm{hom}}(\theta \times c_\theta) + c_\theta = \int_Q F d\mu, \qquad \theta \in \varepsilon^{-1} Q'. \tag{4.12}$$

By setting $\phi = 1$ in (4.7), one infers that

$$\theta \cdot \int_Q F\, d\mu = 0,$$

and therefore $\theta \cdot c_\theta = 0$. This fact will be used in our proof of the convergence estimate.

4.2 Helmholtz decomposition

In the asymptotic analysis of systems of Maxwell equations, the Helmholtz, or Hodge, decomposition [17, Chapter 2], [29, Chapter 9], [49, Section 3.7] for square-integrable functions proves useful. It provides a convenient geometric interpretation of the degeneracy in the problem, namely the fact that the differential expression vanishes on the infinite-dimensional space of gradients of H^2 functions, which suggests representing the relevant L^2 space as an orthogonal sum of curl-free functions with zero mean, divergence-free functions with zero mean and constants. In the present work we require a special version of such a decomposition, which takes into account the quasiperiodicity of the functions involved and also incorporates a class of periodic Borel measures for the underlying L^2 space.

Before formulating the next proposition, we recall that, similarly to the construction of Section 4.1, the notions of a gradient of a quasiperiodic L^2 function with respect to the measure μ and the associated Sobolev space $H^1_\kappa(Q, d\mu)$ can be defined as in Chapter 2.

Proposition 4.2.1. *Suppose that $u \in H^1_\#(Q, d\mu)$. The problem*

$$\overline{e_\kappa}\triangle(e_\kappa \Phi_u) = \overline{e_\kappa}\,\mathrm{div}(e_\kappa u), \tag{4.13}$$

understood in the sense that

$$\int_Q \nabla(e_\kappa\Phi_u)\cdot\overline{\nabla(e_\kappa\phi)}\,d\mu = \int_Q e_\kappa u\cdot\overline{\nabla(e_\kappa\phi)}\,d\mu \qquad \forall\phi\in C^\infty_{\#,0}, \tag{4.14}$$

has a unique scalar solution $\Phi_u \in H^1_{\#,0}$. Here $C^\infty_{\#,0}$ is the set of infinitely smooth Q-periodic functions with zero mean over Q, and $H^1_{\#,0}$ is the set of Q-periodic functions in $H^1_{\mathrm{loc}}(\mathbb{R}^3, d\mu)$ that have zero mean over Q.

Proof. Considering the sesquilinear form on the left hand side of (4.14), the existence and uniqueness of solution Φ_u follows from the Lax-Millgram theorem. Indeed, the continuity of the form is obtained by setting $\nabla(e_\kappa u) = e_\kappa(i\kappa u + \nabla u)$ for all scalar functions $u \in H^1_\#$. The coercivity is a consequence of the Poincaré-type inequality (2.11) proved in Chapter 2 for the scalar setting. □

Using the above statement for each $u \in L^2(Q, d\mu)$ we write

$$u = \widetilde{u} + \int_Q u + \overline{e_\kappa}\nabla(e_\kappa\Phi_u), \tag{4.15}$$

where the function $\widetilde{u}$ satisfies the following conditions on its divergence and mean:

$$\overline{e_\kappa}\,\mathrm{div}\biggl(e_\kappa\biggl(\widetilde{u} + \int_Q u\biggr)\biggr) = 0, \tag{4.16}$$

$$\int_Q \bigl(\widetilde{u} + \overline{e_\kappa}\nabla(e_\kappa\Phi_u)\bigr) = 0. \tag{4.17}$$

The uniqueness part of Proposition 4.2.1 implies that there is a unique function Φ_u (and hence $\widetilde{u}$) such that (4.15) holds.

Summarising the above result, the space of periodic L^2 functions can be written as the orthogonal sum of curl_κ-free functions of the form $\overline{e_\kappa}\nabla(e_\kappa\Phi_u)$, and div_κ-free functions, *cf.* (4.7).

In what follows we make the following assumption about the measure μ.

Assumption 4.2.2. *There exists* $C_{\rm P} > 0$ *such that for all* $\kappa \in Q'$ *and* $(e_\kappa u, \operatorname{curl}(e_\kappa u)) \in H^1_{\operatorname{curl},\kappa}$ *the following Poincaré-type inequality holds:*

$$\left\| u - \int_Q u - \overline{e}_\kappa \nabla(e_\kappa \Phi_u) + \int_Q \overline{e}_\kappa \nabla(e_\kappa \Phi_u) \right\|_{L^2(Q,d\mu)} \leq C_{\rm P} \bigl\| \operatorname{curl}(e_\kappa u) \bigr\|_{L^2(Q,d\mu)}. \qquad (4.18)$$

Remark 4.2.3. *For each fixed* $(e_\kappa u, \operatorname{curl}(e_\kappa u)) \in H^1_{\operatorname{curl},\kappa}$, *denote*

$$\mathfrak{u} := u - \overline{e}_\kappa \nabla(e_\kappa \Phi_u),$$

and notice that $\operatorname{curl}(e_\kappa u)$ *is one of the* κ*-curls of the function* $\mathfrak{u}$ *thus defined, since zero is one of the* κ*-curls of the vector-function* $\overline{e}_\kappa \nabla(e_\kappa \Phi_u)$. *Then one has* $\overline{e}_\kappa \operatorname{div}(e_\kappa \mathfrak{u}) = 0$, *and (4.18) takes the (equivalent) form*

$$\left\| \mathfrak{u} - \int_Q \mathfrak{u} \right\|_{L^2(Q,d\mu)} \leq C_{\rm P} \bigl\| \operatorname{curl}(e_\kappa \mathfrak{u}) \bigr\|_{L^2(Q,d\mu)} \qquad (4.19)$$

In Section 4.3 we show that the following periodic measures satisfy the Poincaré inequality (4.18) (and, equivalently, (4.19)):

(a) Consider a finite set $\{\mathcal{P}_j\}_{j=1}^N$ of planes each of which is orthogonal to one of the coordinate axes and such that $(\cup_{j=1}^N \mathcal{P}_j) \cap Q$ is non-empty and connected. Define the measure μ on Q by the formula

$$\mu(B) = N^{-1} \sum_j |\mathcal{P}_j \cap B|_2 \quad \text{for all Borel } B \subset Q,$$

where $|\cdot|_2$ represents the 2-dimensional Lebesgue measure, *i.e.* $|\mathcal{P}_j \cap B|_2$ is the area of $\mathcal{P}_j \cap B$. In other words, μ is the two-dimensional Hausdorff measure on $(\cup_{j=1}^N \mathcal{P}_j) \cap Q$, normalised by $N = \sum_j |\mathcal{P}_j \cap Q|_2$.

(b) The suitably normalised two-dimensional Hausdorff measure on the intersection with Q of a rigid rotation in $\mathbb{R}^3$ of the union $\cup_{j=1}^N \mathcal{P}_j$ described in a.

(c) The suitably normalised two-dimensional Hausdorff measure on a finite union of sets from the class described in b, under the condition that the union is connected.

(d) The (three-dimensional) Lebesgue measure on Q.

(e) Consider a finite set $\{\mu_j\}_{j=1}^M$ of measures satisfying any of the conditions a, b, d, such that the union of the supports $S_j := \operatorname{supp}(\mu_j)$, $j = 1, \dots, M$, is connected. Define the measure μ by the formula

$$\mu(B) = \frac{\sum_{j=1}^M \mu_j(S_j \cap B)}{\sum_{j=1}^M \mu_j(S_j)} \quad \text{for all Borel } B \subset Q.$$

(Note that c is a particular case of e.)

4.3 Discussion of the validity of (4.18) for some singular measures

Here we show that Assumption 4.2.2 holds for the measures from the class (a) described at the end of Section 4.2, and hence for the classes (b), (c). The validity of Assumption 4.2.2 for the Lebesgue measure (example (d)) is shown easily via an argument based on the Fourier series, see *e.g.* [27].

Consider a finite set $\{\mathcal{P}_j\}_{j=1}^N$ of (two-dimensional) planes in $\mathbb{R}^3$, such that each plane is orthogonal to one of the coordinate axes. Define the measure μ on Q by the formula

$$\mu(B) = N^{-1}\sum_{j=1}^{N} |\mathcal{P}_j \cap B|_j \quad \text{for all Borel } B \subset Q, \tag{4.20}$$

where $|\cdot|_j$ represents the 2-dimensional Lebesgue measure. In what follows, we will use the assumption that $(\cup_{j=1}^N \mathcal{P}_j) \cap Q$ is non-empty and connected. For each $j = 1, \dots, N$, we also consider the measure μ_j defined by

$$\mu_j(B) := |\mathcal{P}_j \cap B|_j \quad \text{for all Borel } B \subset Q,$$

so that $\mu = \sum_{j=1}^N \mu_j$, see (4.20).

4.3.1 Curls of zero for a measure supported by a plane

In this section we fix $j \in \{1, \dots N\}$. In line with Definition 4.1.1, we say that $v \in L^2(Q, d\mu_j)$ is a κ-curl of zero with respect to the measure μ_j if there exists a sequence $\{\phi_n\} \subset [C_\#^\infty]^3$ such that

$$\int_Q |\phi_n|^2 d\mu_j \overset{n\to\infty}{\longrightarrow} 0 \qquad \int_Q \left|\mathrm{curl}(e_\kappa \phi_n) - v\right|^2 d\mu_j \overset{n\to\infty}{\longrightarrow} 0. \tag{4.21}$$

Without loss of generality, we can assume in what follows that the plane $\mathcal{P}_j$ passes through zero and is orthogonal to the x_3 direction.

Proposition 4.3.1. *For each $\kappa \in Q'$, the set of κ-curls of zero with respect to the measure μ_j coincides with*

$$L^2_{\mathrm{s}}(Q, d\mu_j) \oplus L^2_{\mathrm{s}}(Q, d\mu_j) \oplus \{0\},$$

where (see Section 4.2) $L^2_{\mathrm{s}}(Q, d\mu_j)$ is the space of $\mathbb{C}$-valued functions on Q that are square integrable with respect to the measure μ_j.

Proof. For given functions $\xi_1, \xi_2 \in L^2_{\mathrm{s}}(Q, d\mu_j)$, consider sequences of smooth Q-periodic functions, independent of x_3,

$$\{\xi_j^{(n)} = \xi_j^{(n)}(x_1, x_2), \ \ n \in \mathbb{N}\}, \qquad j = 1, 2,$$

such that

$$\xi_j^{(n)} \stackrel{n\to\infty}{\longrightarrow} \xi_j \quad \text{in} \quad L^2_{\rm s}(Q, d\mu_j), \qquad j = 1, 2.$$

Suppose also that functions $\alpha = \alpha(x_3)$, $\beta = \beta(x_3)$ of the single variable x_3 are infinitely smooth and 1-periodic, and that their Taylor expansions at zero have the form $x_3 + O(x_3^2)$. Define

$$\phi_n(x) = \begin{pmatrix} \beta(x_3)\xi_2^{(n)}(x_1, x_2) \\ -\alpha(x_3)\xi_1^{(n)}(x_1, x_2) \\ 0 \end{pmatrix}, \quad x = (x_1, x_2, x_3) \in Q, \qquad n \in \mathbb{N}. \tag{4.22}$$

Then $\{\phi_n\} \subset \big[C^\infty_\#\big]^3$ and by a direct calculation one has, for all $n \in \mathbb{N}$,

$$\overline{e}_\kappa \operatorname{curl}(e_\kappa \phi_n)(x_1, x_2, x_3) = \begin{pmatrix} \big(\alpha'(x_3) - {\rm i}\kappa_3\alpha(x_3)\big)\xi_1^{(n)}(x_1, x_2) \\ \big(\beta'(x_3) + {\rm i}\kappa_3\beta(x_3)\big)\xi_2^{(n)}(x_1, x_2) \\ -\alpha(x_3)(\partial_1 + {\rm i}\kappa_1)\xi_1^{(n)}(x_1, x_2) - \beta(x_3)(\partial_2 + {\rm i}\kappa_2)\xi_2^{(n)}(x_1, x_2) \end{pmatrix},$$

where ∂_j is the operator of differentiation with respect to the variable x_j, $j = 1, 2$. Due of the assumptions on α, β, one has

$$\int_Q |\phi_n|^2 d\mu_j = 0 \qquad \forall n, \tag{4.23}$$

and

$$\overline{e}_\kappa \operatorname{curl}(e_\kappa \phi_n) \stackrel{n\to\infty}{\longrightarrow} (\xi_1, \xi_2, 0)^\top \quad \text{in} \quad L^2(Q, d\mu_j).$$

It follows that $L^2_{\rm s}(Q, d\mu_j) \oplus L^2_{\rm s}(Q, d\mu_j) \oplus \{0\}$ is contained in the set of curls of zero.

On the other hand, any vector of the form

$$(0, 0, \xi_3)^\top, \qquad \xi_3 \in L^2_{\rm s}(Q, d\mu_j),$$

is orthogonal to all κ-curls of zero. Indeed, for any sequence $\xi_3^{(n)} = \xi_3^{(n)}(x_1, x_2)$ of infinitely smooth x_3-independent functions converging to ξ_3 in $L^2_{\rm s}(Q, d\mu_j)$ and any sequence of vector functions

$$\phi^{(n)} = \big(\phi_1^{(n)}, \phi_2^{(n)}, \phi_3^{(n)}\big)^\top \in \big[C_\#^\infty\big]^3, \qquad n \in \mathbb{N},$$

such that

$$\int_Q \big|\phi^{(n)}\big|^2 d\mu_j \stackrel{n\to\infty}{\longrightarrow} 0,$$

one has (due to the fact that the integration by parts is carried out with respect to the variables x_1, x_2 in the plane $\mathcal{P}_j$)

$$\int_Q \big((\partial_2+\mathrm{i}\kappa_2)\phi_1^{(n)}-(\partial_1+\mathrm{i}\kappa_1)\phi_2^{(n)}\big)\overline{\xi_3^{(n)}}d\mu_j = \int_Q \big(\phi_2\overline{(\partial_1+\mathrm{i}\kappa_1)\xi_3^{(n)}}-\phi_1\overline{(\partial_2+\mathrm{i}\kappa_2)\xi_3^{(n)}}\big)d\mu_j = 0. \tag{4.24}$$

It follows from (4.24) that if $\mathrm{curl}(e_\kappa\phi^{(n)}) \to v$ as $n \to \infty$, then ξ_3 is orthogonal to v_3 in $L^2_{\mathrm{s}}(Q, d\mu_j)$, and therefore $(0, 0, \xi_3)^\top$ is orthogonal to v in $L^2(Q, d\mu_j)$. Therefore, the set of κ-curls of zero is contained in $L^2_{\mathrm{s}}(Q, d\mu_j) \oplus L^2_{\mathrm{s}}(Q, d\mu_j) \oplus \{0\}$. This concludes the proof of the claim that these two sets coincide. □

4.3.2 Approximation in $H^1_{\mathrm{curl},\kappa}(Q, d\mu)$ by smooth functions

He we prove the following auxiliary statement, which will allow us to establish (4.19) by first showing that it holds for infinitely smooth functions.

Lemma 4.3.2. *Suppose that $(u, v) \in H^1_{\mathrm{curl},\kappa}(Q, d\mu)$, where the function u is solenoidal (see Section 4.1)*

$$\overline{e}_\kappa \mathrm{div}(e_\kappa u) = 0,$$

and $\mathrm{curl}(e_\kappa u)$ *is pointwise orthogonal to the support of measure μ. Then there exists a sequence $\{\phi_n\} \subset [C_\#^\infty]^3$ such that*

$$\big(e_\kappa\phi_n, \mathrm{curl}(e_\kappa\phi_n)\big) \stackrel{n\to\infty}{\longrightarrow} (u, v) \;\text{ in }\; L^2(Q, d\mu) \oplus L^2(Q, d\mu) \tag{4.25}$$

and the following two properties hold:

$$\overline{e}_\kappa \mathrm{div}(e_\kappa\phi_n) = 0 \tag{4.26}$$

in the sense of (4.7) with $F = \phi_n$, and the vector $\mathrm{curl}(e_\kappa\phi_n)$ *is pointwise orthogonal to* $\mathrm{supp}(\mu)$ *(excluding the lines of intersection of the planes $\mathcal{P}_j$, $j = 1, \dots, N$.)*

Proof. According to Definition 4.1.1, there exists a sequence $\{\widetilde{\phi}_n, n \in \mathbb{N}\} \subset [C_\#^\infty]^3$ approximating (u, v) in the sense that (4.25) holds with ϕ_n replaced by $\widetilde{\phi}_n$, however one does not necessarily have $\overline{e}_\kappa \mathrm{div}(e_\kappa \widetilde{\phi}_n) = 0$. In order to "correct" the sequence $\{\widetilde{\phi}_n\}$, for each n consider the solution $w_n \in H^1_\#$ (see *e.g.* Section 4.2) to the elliptic problem

$$-\Delta w_n = \mathrm{div}\big(e_\kappa \widetilde{\phi}_n\big) \tag{4.27}$$

understood in the weak sense with respect to the measure μ :

$$\int_Q \nabla w_n \cdot \nabla \varphi d\mu = -\int_Q e_\kappa \widetilde{\phi}_n \cdot \nabla \varphi d\mu \qquad \forall \varphi \in C^\infty_\#. \tag{4.28}$$

The problem (4.28) has a unique solution $(w_n, \nabla w_n) \in H^1_\#$, which has the property that ∇w_n is orthogonal to all *gradients of zero* [71] with respect to the measure μ : indeed, setting $\varphi = \varphi_j$, $j \in \mathbb{N}$, in (4.28), where

$$\int_Q |\varphi_j|^2 d\mu \overset{j\to\infty}{\longrightarrow} 0, \qquad \int_Q |\nabla \varphi_j - v|^2 d\mu \overset{j\to\infty}{\longrightarrow} 0 \qquad v \in L^2(Q, d\mu),$$

and passing in the obtained identity to the limit as $j \to \infty$ yields

$$\int_Q \nabla w_n \cdot v \, d\mu = 0,$$

as claimed. Following an argument similar to that given in [71, Section 3.1], see also [72, Section 4], it is shown that the set of gradients of zero is a closed subspace of $L^2(Q, d\mu)$ consisting of vector functions that, when restricted to the plane $\mathcal{P}_j$, $j = 1, \dots, N$, are pointwise orthogonal to it. Furthermore, it is straightforward to show that for each n the function w_n is infinitely smooth on $Q \cap \mathcal{P}_j$, *e.g.* by deducing the decay properties of the coefficients of its Fourier series with respect to x_1, x_2 in terms of the decay properties, as $n \to \infty$, of the Fourier coefficients of $\widetilde{\phi}_n$.

For each $n \in \mathbb{N}$, we consider an infinitely smooth function $\widetilde{w}_n$ on Q that for each $j \in \{1, \dots, N\}$ coincides with w_n on $Q \cap \mathcal{P}_j$ and has zero gradient in the variables orthogonal to $\mathcal{P}_j$. (Such a smooth extension from $(\cup_{j=1}^N \mathcal{P}_j) \cap Q$ to Q can be obtained in a standard way by an appropriate partition of unity on Q, carrying out standard extensions in corner, edge, and face regions, and using appropriate mollifiers.) Clearly, on $\mathrm{supp}(\mu)$ one has

$$\mathrm{curl}\big(e_\kappa(\overline{e}_\kappa \nabla \widetilde{w}_n)\big) = \mathrm{curl}\,(\nabla \widetilde{w}_n) = \mathrm{curl}\,(\nabla w_n) = 0. \tag{4.29}$$

Furthermore, writing (4.28) in the form (where we take advantage of u being solenoidal)

$$\int_Q \nabla w_n \cdot \nabla\varphi d\mu = \int_Q e_\kappa(u - \widetilde{\phi}_n)\cdot\nabla\varphi d\mu \qquad \forall\varphi \in C^\infty_\#,$$

setting $\varphi = w_n$, and using the fact that the right-hand side of the result goes to zero as $n \to 0$, we obtain

$$\int_Q \bigl|\nabla\widetilde{w}_n\bigr|^2 d\mu = \int_Q |\nabla w_n|^2 d\mu \overset{n\to\infty}{\longrightarrow} 0.$$

Combining this observation with (4.29) and (4.27), we conclude that the functions

$$\widehat{\phi}_n := \widetilde{\phi}_n + \overline{e}_\kappa\nabla\widetilde{w}_n, \qquad n\in\mathbb{N},$$

are smooth and have the convergence properties

$$\int_Q \bigl|\widehat{\phi}_n - u\bigr|^2 d\mu \overset{n\to\infty}{\longrightarrow} 0, \qquad \int_Q \bigl|\mathrm{curl}\bigl(e_\kappa\widehat{\phi}_n\bigr) - v\bigr|^2 d\mu \overset{n\to\infty}{\longrightarrow} 0,$$

and $e_\kappa\widehat{\phi}_n$ is solenoidal for each $n\in\mathbb{N}$, as required in (4.25), (4.26).

In order to fulfil the second property claimed in the lemma, we construct a further "correction" to the sequence $\{\widetilde{\phi}_n\}$, which does not affect the properties (4.25), (4.26). For each $j\in\{1,\dots N\}$, consider the rotation R_j in $\mathbb{R}^3$ such that the plane $R_j\mathcal{P}_j$ passes through zero and is orthogonal to the x_3 direction. To simplify the notation, we fix j and assume, as in Section 4.3.1, that $R_j = I$.

Under the above convention, notice that the projection of $\mathrm{curl}(e_\kappa\widehat{\phi}_n)$ onto the plane $\mathcal{P}_j$, restricted to the set $\mathcal{P}_j\cap Q$ (*i.e.* the support of μ_j) is a smooth function

$$\psi_n = e_\kappa\bigl((\psi_n)_1(x_1,x_2),(\psi_n)_2(x_1,x_2),0\bigr)^\top, \quad (x_1,x_2)\in[0,1)^2.$$

where, using the notation $\widehat{\phi}_n = ((\widehat{\phi}_n)_1,(\widehat{\phi}_n)_2,(\widehat{\phi}_n)_3)$,

$$\begin{aligned}(\psi_n)_1 &= \bigl((\mathrm{i}\kappa_2+\partial_2)(\widehat{\phi}_n)_3 - (\mathrm{i}\kappa_3+\partial_3)(\widehat{\phi}_n)_2\bigr)\big|_{x_3=0},\\ (\psi_n)_2 &= \bigl((\mathrm{i}\kappa_3+\partial_3)(\widehat{\phi}_n)_1 - (\mathrm{i}\kappa_1+\partial_1)(\widehat{\phi}_n)_3\bigr)\big|_{x_3=0}.\end{aligned} \tag{4.30}$$

Consider the vector (*cf.* (4.22))

$$\widehat{\psi}_n(x) = \begin{pmatrix}\beta(x_3)(\psi_n)_1(x_1,x_2)\\ -\alpha(x_3)(\psi_n)_2(x_1,x_2)\\ 0\end{pmatrix}, \quad x=(x_1,x_2,x_3)\in Q, \qquad n\in\mathbb{N},$$

where the functions $\alpha = \alpha(x_3)$, $\beta = \beta(x_3)$ of the single variable x_3 are infinitely smooth and 1-periodic, and that their Taylor expansions at zero have the form $x_3 + O(x_3^2)$. Similarly to the argument (4.29), we notice that $\operatorname{curl}(e_\kappa \widehat{\psi}_n) = \psi_n$, now viewed as a function on the whole of Q. Furthermore, the vector $\widehat{\psi}_n$ is trivially solenoidal, as the vector $\widehat{\psi}_n$ vanishes on $\mathcal{P}_j \cap Q$, and $\operatorname{curl}(e_\kappa \widehat{\psi}_n) \to 0$ in $L^2(Q, d\mu)$ as $n \to \infty$, due to the assumption of pointwise orthogonality of $\operatorname{curl}(e_\kappa \mathfrak{u})$ to $\mathcal{P}_j \cap Q$.

The above construction is repeated for each $j \in \{1, \dots, N\}$, now taking into account that it will be preceded by the rotation R_j. Relabel by $\widehat{\psi}_n^{(j)}$ the elements of the constructed sequence $\widehat{\psi}$. As a result, the sequence

$$\phi_n = \widehat{\phi}_n - \sum_{j=1}^{N} R_j^\top \widehat{\psi}_n^{(j)}, \qquad n \in \mathbb{N}$$

satisfies all the required conditions. $\square$

4.3.3 Poincaré-type inequality

In this section we carry out the proof of the Poincaré-type inequality (*cf.* (4.19))

$$\left\| \mathfrak{u} - \int_Q \mathfrak{u} \right\|_{L^2(Q, d\mu_j)} \leq C_{\mathrm{P}} \left\| \operatorname{curl}(e_\kappa \mathfrak{u}) \right\|_{L^2(Q, d\mu_j)} \tag{4.31}$$

for the measure defined by (4.20).

Notice that in the inequality (4.31) we can assume, without loss of generality, that the vector $\operatorname{curl}(e_\kappa \mathfrak{u})$ is orthogonal to $\mathcal{P}_j$ at almost every point of $\mathcal{P}_j \cap Q$. Indeed, one can write

$$\operatorname{curl}(e_\kappa \mathfrak{u}) = w_1 + w_2,$$

where w_2 is the projection of $\operatorname{curl}(e_\kappa \mathfrak{u})$ onto the subspace of $L^2(Q, d\mu)$ consisting of κ-curls of zero, w_1 is another value of the κ-curl of $\mathfrak{u}$, so that w_1 and w_2 are orthogonal in the sense of $L^2(Q, d\mu)$. As we showed in Section 4.3.1, in the case of the measure μ_j κ-curls of zero are parallel to $\mathcal{P}_j$ at each point, so w_1 is pointwise parallel to $\mathcal{P}_j$ and w_2 is pointwise orthogonal to $\mathcal{P}_j$. In what follows we can therefore assume that $\operatorname{curl}(e_\kappa \mathfrak{u})$ is orthogonal to $\mathcal{P}_j$. This will allow us, in particular, to use Lemma 4.3.2.

We first prove an auxiliary proposition reflecting the vectorial nature of the inequality (4.31), due to the presence of the operator curl and then combine it with the "scalar" Poincaré inequality applied to each component of the vector $\mathfrak{u}$.

Having proved (4.19) with the measure μ replaced by μ_j, we will then show that it holds for μ as well (possibly with a larger constant C_{P}), using the assumption that the the set

$$(\cup_{j=1}^N \mathcal{P}_j) \cap Q = \cup_{j=1}^N (\mathcal{P}_j \cap Q)$$

is connected.

The norm of the transversal curl is the norm of the tangential gradient

Here we fix $j \in \{1, \dots, N\}$ and, as in Section 4.3.1, assume that the plane $\mathcal{P}_j$ passes through zero and is orthogonal to the x_3 direction. For $\kappa \in Q'$ and a function $\phi \in [C^\infty_\#]^3$, we denote by $\widetilde{\nabla}(e_\kappa\phi)$ the pointwise orthogonal projection of the $\nabla(e_\kappa\phi)$ onto the (x_1, x_2)-plane.

Proposition 4.3.3. *Suppose that a vector function $\phi \in [C^\infty_\#]^3$ is solenoidal, i.e. (cf. (4.26))*

$$\overline{e}_\kappa \mathrm{div}(e_\kappa\phi) = 0 \tag{4.32}$$

and that the vector $\mathrm{curl}(e_\kappa\phi)$ *is pointwise parallel to x_3 at each point of $\mathcal{P}_j \cap Q$. Then, for all $\kappa \in Q'$, one has*

$$\left\|\widetilde{\nabla}(e_\kappa\phi)\right\|^2_{L^2(Q,d\mu_j)} = \left\|\mathrm{curl}(e_\kappa\phi)\right\|^2_{L^2(Q,d\mu_j)}, \tag{4.33}$$

Proof. We we expand the function ϕ into the standard Fourier series:

$$\phi(x) = \sum_{l\in\mathbb{Z}^3} \exp(2\pi \mathrm{i} l\cdot x)c_l, \quad x \in Q, \qquad c_l \in \mathbb{C}^3,\ l \in \mathbb{Z}^3 \tag{4.34}$$

and notice that

$$\mathrm{curl}(e_\kappa\phi_n)(x) = \mathrm{i}e_\kappa \sum_{l\in\mathbb{Z}^3} \exp(\mathrm{i}2\pi l\cdot x)\big(c_l \times (\kappa + 2\pi l)\big), \quad x \in Q,$$

where the series converges in the norm of $L^2(Q)$, with respect to the Lebesgue measure on Q. Since $\mathrm{curl}(e_\kappa\phi)$ is pointwise orthogonal to $\mathcal{P}_j$, it follows that for each $l \in \mathbb{Z}^3$ the vector $c_l \times (\kappa + 2\pi l)$ is orthogonal to $\mathcal{P}_j$, *i.e.* it is parallel to the x_3 direction.

For each $x \in Q$, we denote $(x_1, x_2) =: \widetilde{x}$. and, similarly, for each value $\kappa \in Q'$ of the quasimomentum, we denote $\widetilde{\kappa} := (\kappa_1, \kappa_2)$. Finally, for each "multi-index" $l \in \mathbb{Z}^3$, we consider the "sub-index" $\widetilde{l} := (l_1, l_2) \in \mathbb{Z}^2$. We write finite truncations of (4.34) in the form ($K \in \mathbb{N}$)

$$\phi_K(x) = \sum_{|l|\le K} c_l \exp(2\pi \mathrm{i} l\cdot x) = \sum_{|\widetilde{l}|\le K} \ \sum_{|l_3|\le K-|\widetilde{l}|} c_{\widetilde{l},l_3} \exp\big(2\pi \mathrm{i}(\widetilde{l}, l_3)\cdot(\widetilde{x}, x_3)\big), \quad x \in Q. \tag{4.35}$$

In the remainder of this section, for brevity, we omit the summation ranges for $\widetilde{l}$, l_3, which are the same as in (4.35) throughout. From (4.35) one has

$$\int_Q \left|\mathrm{i}\varphi\kappa + \nabla\phi_K\right|^2 d\mu_j = \sum_{\widetilde{l}} \biggl(\sum_{l_3} \Bigl\{ c_{(\widetilde{l},l_3)} \otimes \bigl(\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3\bigr) \Bigr\} \biggr)^\top \times \biggl(\sum_{m_3} \Bigl\{ \overline{c}_{(\widetilde{l},m_3)} \otimes \bigl(\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3\bigr) \Bigr\} \biggr). \tag{4.36}$$

Rearranging the product under the external summation in (4.36) yields[3]

$$\int_Q \left|\mathrm{i}\varphi\kappa + \nabla\phi_K\right|^2 d\mu_j = \sum_{\widetilde{l}} \sum_{l_3,m_3} \bigl\{ c_{(\widetilde{l},l_3)} \cdot c_{(\widetilde{l},m_3)} \bigr\} \bigl\{ (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) \bigr\}. \tag{4.37}$$

In order to manipulate the above expression into a convenient form, we notice two properties of the Fourier series for ϕ, due to the assumptions that it is solenoidal (see (4.32)) and that its κ-curl is orthogonal to $\mathcal{P}_j$ (*i.e.* parallel to the x_3-direction). In terms of the Fourier coefficients c_l, the first condition can be written as follows:

$$\begin{aligned} 0 &= \int_Q \sum_{\widetilde{p},l_3} \exp\bigl(\mathrm{i}(\widetilde{p}\cdot\widetilde{x} + l_3x_3)\bigr) c_{(\widetilde{p},l_3)} \cdot \exp\bigl(-\mathrm{i}(\widetilde{l}\cdot\widetilde{x} + m_3x_3)\bigr)(\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) d\mu_j \\ &= \int_{(0,1)^2} \sum_{\widetilde{p},l_3} \exp(\mathrm{i}\widetilde{p}\cdot\widetilde{x}) c_{(\widetilde{p},l_3)} \cdot \exp(-\mathrm{i}\widetilde{l}\cdot\widetilde{x})(\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) d\widetilde{x} \\ &= \sum_{l_3} c_{(\widetilde{l},l_3)} \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) \qquad \forall\, \widetilde{l} \in \mathbb{Z}^2,\ m_3 \in \mathbb{Z}, \end{aligned} \tag{4.38}$$

which is obtained by setting

$$\varphi(x) = \exp\bigl(-\mathrm{i}\widetilde{l}\cdot\widetilde{x} + m_3x_3\bigr), \quad x \in Q, \qquad \widetilde{l} \in \mathbb{Z}^2,\ m_3 \in \mathbb{Z},$$

as the test function for (4.32).

Similarly, the second condition takes the form

$$\begin{aligned} 0 &= \int_Q \sum_{\widetilde{p},l_3} \exp(\mathrm{i}\widetilde{p}\cdot\widetilde{x}) \bigl(c_{(\widetilde{p},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{p}, \kappa_3 + 2\pi l_3) \bigr) \cdot \exp(-\mathrm{i}\widetilde{l}\cdot x) a, \\ &= \sum_{l_3} \Bigl(c_{(\widetilde{l},l_3)} \times \bigl(\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3\bigr) \Bigr) \cdot a \qquad \forall\, \widetilde{l} \in \mathbb{Z}^2,\ a \in (0,1)^2. \end{aligned} \tag{4.39}$$

[3]Recall that by $a \cdot b$ we denote the sesquilinear inner product of $a, b \in \mathbb{C}^3$.

Using standard formulae of vector calculus, we write, for each $\widetilde{l} \in \mathbb{Z}^2$, $m_3 \in \mathbb{Z}$,

$$
\begin{aligned}
&\sum_{l_3} \Big(c_{(\widetilde{l},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \Big) \cdot \Big(c_{(\widetilde{l},m_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) \Big) \\
&= \sum_{l_3} c_{(\widetilde{l},m_3)} \cdot \Big\{ (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) \times \Big(c_{(\widetilde{l},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \Big) \Big\} \\
&= \sum_{l_3} c_{(\widetilde{l},m_3)} \cdot \Big\{ (\widetilde{\kappa} + 2\pi\widetilde{l}, 0) \times \Big(c_{(\widetilde{l},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \Big) \Big\} \\
&\qquad\qquad + c_{(\widetilde{l},m_3)} \cdot \Big\{ (0, \kappa_3 + 2\pi m_3) \times \sum_{l_3} \Big(c_{(\widetilde{l},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \Big) \Big\} \\
&= \sum_{l_3} c_{(\widetilde{l},m_3)} \cdot \Big\{ (\widetilde{\kappa} + 2\pi\widetilde{l}, 0) \times \Big(c_{(\widetilde{l},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \Big) \Big\} \\
&= \sum_{l_3} \Big(\big\{ c_{(\widetilde{l},l_3)} \cdot c_{(\widetilde{l},m_3)} \big\} \big\{ (\widetilde{\kappa} + 2\pi\widetilde{l}, 0) \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}, 0) \big\} \\
&\qquad\qquad - \big\{ c_{(\widetilde{l},l_3)} \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) \big\} \big\{ \overline{c}_{(\widetilde{l},m_3)} \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}, 0) \big\} \Big) \\
&= \sum_{l_3} \big\{ c_{(\widetilde{l},l_3)} \cdot c_{(\widetilde{l},m_3)} \big\} \big\{ (\widetilde{\kappa} + 2\pi\widetilde{l}) \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}) \big\} \\
&\qquad\qquad - \Big\{ \sum_{l_3} c_{(\widetilde{l},l_3)} \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) \Big\} \big\{ \overline{c}_{(\widetilde{l},m_3)} \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}, 0) \big\} \\
&= \sum_{l_3} \big\{ c_{(\widetilde{l},l_3)} \cdot c_{(\widetilde{l},m_3)} \big\} \big\{ (\widetilde{\kappa} + 2\pi\widetilde{l}) \cdot (\widetilde{\kappa} + 2\pi\widetilde{l}) \big\}.
\end{aligned}
$$

Here, for the third equality we use the fact that by (4.39), the vector

$$\sum_{l_3} \Big(c_{(\widetilde{l},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \Big)$$

is orthogonal to the (x_1, x_2)-plane and hence parallel to the vector $(0, \kappa_3 + 2\pi m_3)$, and for the sixth equality we use (4.38). It follows that

$$
\begin{aligned}
&\int_Q \big| \nabla (e_\kappa \phi) \big|^2 d\mu_j = \lim_{K \to \infty} \int_Q \big| \mathrm{i} \varphi \kappa + \nabla \phi_K \big|^2 d\mu_j \\
&= \lim_{K \to \infty} \sum_{|\widetilde{l}| \le K} \sum_{\substack{|l_3| \le K - |\widetilde{l}| \\ |m_3| \le K - |\widetilde{l}|}} \Big\{ \big\{ c_{(\widetilde{l},l_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi l_3) \big\} \cdot \big\{ c_{(\widetilde{l},m_3)} \times (\widetilde{\kappa} + 2\pi\widetilde{l}, \kappa_3 + 2\pi m_3) \big\}
\end{aligned}
$$

$$+\left\{c_{(\widetilde{l},l_3)}\cdot c_{(\widetilde{l},m_3)}\right\}(\kappa_3+2\pi l_3)\cdot(\kappa_3+2\pi m_3)\Biggr\}$$

$$=\lim_{K\to\infty}\left\{\int_Q\bigl|(\mathrm{i}\kappa+\nabla)\times\phi_K\bigr|^2d\mu_j+\int_Q\bigl|(\mathrm{i}\kappa_3+\partial_3)\phi_K\bigr|^2d\mu_j\right\}$$

$$=\lim_{K\to\infty}\left\{\int_Q\bigl|\mathrm{curl}(e_\kappa\phi_K)\bigr|^2d\mu_j+\int_Q\bigl|\partial_3(e_\kappa\phi_K)\bigr|^2d\mu_j\right\}=\int_Q\bigl|\mathrm{curl}(e_\kappa\phi)\bigr|^2d\mu_j+\int_Q\bigl|\partial_3(e_\kappa\phi)\bigr|^2d\mu_j,$$

and therefore

$$\bigl\|\widetilde{\nabla}(e_\kappa\phi)\bigr\|^2_{L^2(Q,d\mu_j)}=\bigl\|\nabla(e_\kappa\phi)\bigr\|^2_{L^2(Q,d\mu_j)}-\bigl\|\partial_3(e_\kappa\phi)\bigr\|^2_{L^2(Q,d\mu_j)}=\bigl\|\mathrm{curl}(e_\kappa\phi)\bigr\|^2_{L^2(Q,d\mu_j)},$$

as required. $\square$

"Scalar" Poincaré inequality for a single plane

We continue working with a fixed $j\in\{1,\dots,N\}$ and assume, without loss of generality, that the plane $\mathcal{P}_j$ passes through zero and is orthogonal to the x_3-direction. For a function $\phi\in C^\infty_\#$, we denote by $\widetilde{\nabla}\phi(x)\in\mathbb{R}^2$, $x\in Q$, the (pointwise) projection of its gradient onto the (x_1,x_2)-plane.

We write

$$\phi(\widetilde{x})-\int_Q\phi d\mu_j=\sum_{\widetilde{l}\in\mathbb{Z}^2\setminus\{0\}}c_{\widetilde{l}}\exp\bigl(2\pi\mathrm{i}\widetilde{l}\cdot\widetilde{x}\bigr),\quad \widetilde{x}\in[0,1)^2,\qquad c_{\widetilde{l}}\in\mathbb{C},\quad \widetilde{l}\in\mathbb{Z}^2\setminus\{0\},$$

and notice that, for each $\widetilde{\kappa}\in[-\pi,\pi)^2$, one has, assuming ϕ is non-constant on $\mathcal{P}_j\cap Q$,

$$\left(\int_Q\left|\phi-\int_Q\phi d\mu_j\right|^2d\mu_j\right)^{-1}\int_Q\bigl|\mathrm{i}\varphi\widetilde{\kappa}+\widetilde{\nabla}\phi\bigr|^2d\mu_j$$

$$=\left(\sum_{\widetilde{l},\widetilde{m}\in\mathbb{Z}^2\setminus\{0\}}\alpha_{\widetilde{l}\widetilde{m}}c_{\widetilde{l}}\overline{c_{\widetilde{m}}}\right)^{-1}\left(\sum_{\widetilde{l},\widetilde{m}\in\mathbb{Z}^2\setminus\{0\}}\alpha_{\widetilde{l}\widetilde{m}}c_{\widetilde{l}}\overline{c_{\widetilde{m}}}(\widetilde{\kappa}+2\pi\widetilde{l})\cdot(\widetilde{\kappa}+2\pi\widetilde{m})\right),$$

where

$$\alpha_{\widetilde{l}\widetilde{m}}:=\int_{(0,1)^2}\exp\bigl(2\pi\mathrm{i}(\widetilde{l}-\widetilde{m})\cdot\widetilde{x}\bigr)d\widetilde{x}=\begin{cases}1, & \widetilde{l}=\widetilde{m},\\ 0 & \text{otherwise.}\end{cases}$$

It follows that

$$\left(\int_Q\left|\phi-\int_Q\phi d\mu_j\right|^2d\mu_j\right)^{-1}\int_Q\bigl|\mathrm{i}\varphi\widetilde{\kappa}+\widetilde{\nabla}\phi\bigr|^2d\mu_j=\left(\sum_{\widetilde{l}\in\mathbb{Z}^2\setminus\{0\}}|c_l|^2\right)^{-1}\left(\sum_{\widetilde{l},\widetilde{m}\in\mathbb{Z}^2\setminus\{0\}}|c_l|^2|\widetilde{\kappa}+2\pi\widetilde{l}|^2\right)\geq\pi^2,$$

and hence

$$\int_Q \left|\phi - \int_Q \phi d\mu_j\right|^2 d\mu_j \le \pi^{-2}\big\|\widetilde{\nabla}(e_\kappa\phi)\big\|^2_{L^2(Q,d\mu_j)}, \tag{4.40}$$

where $\widetilde{\nabla}(e_\kappa\phi)$ is the "tangential" gradient introduced above. If the function ϕ is constant on $\mathcal{P}_j \cap Q$, the inequality (4.40) is satisfied trivially. Note also that an inequality of the same for as (4.40) has thus been established for vector functions $\phi \in [C_\#]^3$, by applying it component-wise and adding the inequalities obtained for the individual components. Below we discuss the vector case, for which the Poincaré inequality for any of the measures μ_j, $j = 1, \dots, N$, looks the same as (4.40), where ϕ is now a smooth vector function.

Connectivity argument

For the measure $\mu = \sum_{j=1}^N \mu_j$ and $\phi \in [C^\infty_\#]^3$, we denote by $\widetilde{\nabla}(e_\kappa\phi)$ the (component-wise) tangential gradient of ϕ at points of $\mathrm{supp}(\mu)$, *i.e.* the orthogonal projection of $\nabla(e_\kappa\phi)$ onto $\mathrm{supp}(\mu)$.

Suppose that for $j, k \in \{1, \dots, N\}$ the planes $\mathcal{P}_j$ and $\mathcal{P}_k$ intersect and fix a point $\alpha_{jk} \in \mathcal{P}_j \cap \mathcal{P}_k \cap Q$. For any $\kappa \in Q'$, any function $\phi \in [C^\infty_\#]^3$, and all $x \in \mathcal{P}_j \cap Q$, $y \in \mathcal{P}_k \cap Q$, one has

$$\begin{aligned} e_\kappa(x)\phi(x) - e_\kappa(y)\phi(y) = &\int_0^1 \nabla(e_\kappa\phi)\big(\alpha_{jk} + t(x - \alpha_{jk})\big)dt \cdot (x - \alpha_{jk}) \\ &- \int_0^1 \nabla(e_\kappa\phi)\big(\alpha_{jk} + t(y - \alpha_{jk})\big)dt \cdot (y - \alpha_{jk}). \end{aligned} \tag{4.41}$$

Multiplying both sides of (4.41) by $e_\kappa(y)^{-1} = e_\kappa(-y)$ and integrating over $y \in Q$ with respect to the measure μ_k (recalling that $\mathrm{supp}(\mu_k) = \mathcal{P}_k \cap Q$) yields

$$\begin{aligned} e_\kappa(x)\phi(x)\int_Q e_\kappa^{-1} d\mu_k - \int_Q \phi d\mu_k \le \sqrt{2}\bigg(&\int_0^1 \big|\widetilde{\nabla}(e_\kappa\phi)\big(\alpha_{jk} + t(x - \alpha_{jk})\big)\big|dt \\ &+ \big\|\widetilde{\nabla}(e_\kappa\phi)\big\|_{L^2(Q,d\mu_k)}\bigg) \qquad \forall x \in \mathcal{P}_j \cap Q. \end{aligned} \tag{4.42}$$

Furthermore, multiplying both sides of (4.42) by $e_\kappa(x)^{-1}$ and integrating over $x \in Q$ with respect to the measure μ_j yields

$$\int_Q \phi d\mu_j - \int_Q \phi d\mu_k \le \sqrt{2}\Big(\big\|\widetilde{\nabla}(e_\kappa\phi)\big\|_{L^2(Q,d\mu_j)} + \big\|\widetilde{\nabla}(e_\kappa\phi)\big\|_{L^2(Q,d\mu_k)}\Big) \le \sqrt{2}\big\|\widetilde{\nabla}(e_\kappa\phi)\big\|_{L^2(Q,d\mu)}.$$

By interchanging k and j if necessary, we thus obtain

$$\left|\int_Q \phi d\mu_j - \int_Q \phi d\mu_k\right| \le \sqrt{2}\big\|\widetilde{\nabla}(e_\kappa\phi)\big\|_{L^2(Q,d\mu)}.$$

Next, notice that since $(\cup_{j=1}^N \mathcal{P}_j) \cap Q$ is connected by assumption, for each pair of planes in the union there is a "path" from one plane to the other involving at most N planes, such that any "adjacent" planes in the path intersect. It follows that for all pairs j, k the following bound holds:

$$\left| \int_Q \phi d\mu_j - \int_Q \phi d\mu_k \right| \leq \sqrt{2} N \big\| \widetilde{\nabla}(e_\kappa \phi) \big\|_{L^2(Q,d\mu)}. \tag{4.43}$$

Finally, using (4.43) and standard arithmetic inequalities, we obtain

$$\begin{aligned}
\int_Q \left| \phi - \int_Q \phi \right|^2 d\mu &= \sum_{j=1}^N \int_Q \left| \phi - \int_Q \phi \right|^2 d\mu_j = \sum_{j=1}^N \int_Q \left| \phi - \sum_{k=1}^N N^{-1} \int_Q \phi d\mu_k \right|^2 d\mu_j \\
&= \sum_{j=1}^N \int_Q \left| \sum_{k=1}^N N^{-1} \Big(\phi - \int_Q \phi d\mu_k \Big) \right|^2 d\mu_j \leq \sum_{j=1}^N \sum_{k=1}^N N^{-1} \int_Q \left| \Big(\phi - \int_Q \phi d\mu_k \Big) \right|^2 d\mu_j \\
&= \sum_{j=1}^N \sum_{k=1}^N N^{-1} \int_Q \left| \phi - \int_Q \phi d\mu_j + \Big(\int_Q \phi d\mu_j - \int_Q \phi d\mu_k \Big) \right|^2 d\mu_j \\
&\leq 2 \sum_{j=1}^N \sum_{k=1}^N N^{-1} \left\{ \int_Q \left| \phi - \int_Q \phi d\mu_j \right|^2 d\mu_j + 2N^2 \| \widetilde{\nabla}(e_\kappa \phi) \|^2_{L^2(Q,d\mu)} \right\} \\
&= 2 \sum_{j=1}^N \left\{ \int_Q \left| \phi - \int_Q \phi d\mu_j \right|^2 d\mu_j + 2N^2 \| \widetilde{\nabla}(e_\kappa \phi) \|^2_{L^2(Q,d\mu)} \right\} \\
&\leq 2 \sum_{j=1}^N \left\{ \pi^{-2} \| \widetilde{\nabla}(e_\kappa \phi) \|^2_{L^2(Q,d\mu_j)} + 2N^2 \| \widetilde{\nabla}(e_\kappa \phi) \|^2_{L^2(Q,d\mu)} \right\} \\
&\leq 2 \big(\pi^{-2} + 2N^3 \big) \| \widetilde{\nabla}(e_\kappa \phi) \|^2_{L^2(Q,d\mu)}.
\end{aligned}$$

Combining the above bound and the result of Proposition 4.3.3 applied for each $j = 1, \dots, N$, we obtain

$$\int_Q \left| \phi - \int_Q \phi \right|^2 d\mu \leq C_{\mathrm{P}} \big\| \mathrm{curl}(e_\kappa \phi) \big\|^2_{L^2(Q,d\mu)}, \tag{4.44}$$

with

$$C_{\mathrm{P}} = 2 \big(\pi^{-2} + 2N^3 \big), \tag{4.45}$$

which we note depends on N only.

According to the result of Section 4.3.2, the pair $(\mathfrak{u}, \mathrm{curl}(e_\kappa \mathfrak{u}))$ is approximated by functions $\phi_n \in [C^\infty_\#]^3$ satisfying the conditions of Proposition 4.3.3, *i.e.* such that

$$\int_Q e_\kappa \phi_n \cdot \nabla(e_\kappa \psi)\, d\mu = 0 \qquad \forall \psi \in C^\infty_\#$$

and $\operatorname{curl}(e_\kappa \phi_n)$ is pointwise orthogonal to $\operatorname{supp}(\mu)$, where the approximation is understood in the sense that (*cf.* (4.25))

$$\bigl(e_\kappa \phi_n, \operatorname{curl}(e_\kappa \phi_n)\bigr) \overset{n\to\infty}{\longrightarrow} \bigl(\mathfrak{u}, \operatorname{curl}(e_\kappa \mathfrak{u})\bigr) \ \text{ in } \ L^2(Q, d\mu) \oplus L^2(Q, d\mu).$$

Writing the bound (4.44) with $\phi = \phi_n$, where $\{\phi_n\} \subset [C^\infty_\#]^3$ is the approximating sequence for u as described above, and passing to the limit as $n \to \infty$ yields the inequality (4.19) with the constant C_{P} given by (4.45).

4.4 Asymptotic approximation of u^ε_θ

In order to write an asymptotic expansion for the solution u^ε_θ of (4.11), we consider the "cell problem" (*cf.* [27])

$$\operatorname{curl}(A \operatorname{curl} \widetilde{N}) = -\operatorname{curl} A, \qquad \operatorname{div} \widetilde{N} = 0, \qquad \int_Q \widetilde{N} d\mu = 0, \tag{4.46}$$

where $(\operatorname{curl} \widetilde{N})_{ij} = \varepsilon_{ist} N_{tj,s}$ and $(\operatorname{div} N)_i = N_{si,s}$ for $i, j, s, t \in \{1, 2, 3\}$ following the Levi-Civita notation. The first equation is understood in the sense of the integral identity

$$\int_Q A \operatorname{curl} \widetilde{N} \cdot \overline{\operatorname{curl} \phi}\, d\mu = -\int_Q A\, \overline{\operatorname{curl} \phi}\, d\mu \qquad \forall \phi \in C^\infty_\#. \tag{4.47}$$

Proposition 4.4.1. *There exists a unique matrix function $\widetilde{N}$ with columns in $H^1_{\mathrm{curl}}(Q, d\mu)$, solution to* (4.46)*.*

Proof. It follows from the inequality (4.18) with $\kappa = 0$ that the skew-symmetric sesquilinear form

$$\int_Q A \operatorname{curl} u \cdot \overline{\operatorname{curl} v}\, d\mu, \qquad u, v \in H^1_{\mathrm{curl}}(Q, d\mu) \cap \Bigl\{ u : \operatorname{div} u = 0, \int_Q u = 0 \Bigr\},$$

is coercive. Noting also that it is also clearly continuous, the claim follows by the Riesz representation theorem. $\square$

Theorem 4.4.2. *Suppose a measure μ such that the Poincaré-type inequality* (4.18) *holds. Furthermore, assume a real-valued matrix function A that is Q-periodic, symmetric, bounded and uniformly positive. Then the following estimate holds for the solutions* *to* (4.11) *with a constant* $C > 0$ *independent of* ε, θ, F*:*

$$\|u_\theta^\varepsilon - c_\theta\|_{L^2(Q,d\mu)} \leq C\varepsilon\|F\|_{L^2(Q,d\mu)}, \tag{4.48}$$

where c_θ *is the vector solution of the homogenised problem* (4.12)*, that is*

$$c_\theta = c_\theta(F) = (\mathfrak{A}_\theta^{\mathrm{hom}} + I)^{-1} \int_Q F\, d\mu. \tag{4.49}$$

Here $\mathfrak{A}_\theta^{\mathrm{hom}}$ *is the matrix-valued quadratic form given in the equation* (4.12)*, with*

$$A^{\mathrm{hom}} := \int_Q A(\operatorname{curl} \widetilde{N} + I)\, d\mu. \tag{4.50}$$

Corollary 4.4.3. *There exists* $C > 0$ *independent of* ε *and of the choice of the sequence* $f^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$*, such that*

$$\|u^\varepsilon - u_{\mathrm{hom}}^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \leq C\varepsilon\|f^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}, \tag{4.51}$$

where u^ε *is the solution of the original problem* (4.1)*, and* $u_{\mathrm{hom}}^\varepsilon$ *is the solution of the homogenised equation* (4.3)*,* (4.50)*.*

Proof Corollary 4.4.3. Throughout the proof we shall drop the superscript ε in f^ε for brevity. For each element of the sequence $f = f^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, consider the Q-periodic function $f_\theta^\varepsilon := \overline{e_{\varepsilon\theta}}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon f$, *cf.* (4.9), so that

$$\int_Q f_\theta^\varepsilon d\mu = \widehat{f}(\theta,\varepsilon), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \widehat{f}(\theta,\varepsilon) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} f\overline{e_\theta}d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.$$

Consider u_θ^ε, solution of (4.11) with $F = f_\theta^\varepsilon$. Using Proposition 4.1.3, we can write the difference between the solutions u^ε and $u_{\mathrm{hom}}^\varepsilon$ to (4.1) and (4.3), respectively, as

$$\begin{aligned} u^\varepsilon - u_{\mathrm{hom}}^\varepsilon &= (\mathcal{A}^\varepsilon + I)^{-1}f - (\mathcal{A}^{\mathrm{hom}} + I)^{-1}f \\ &= \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa(\varepsilon^{-2}\mathcal{A}_{\varepsilon\theta} + I)^{-1}f_\theta^\varepsilon - (\mathcal{A}^{\mathrm{hom}} + I)^{-1}f = \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa u_\theta^\varepsilon - (\mathcal{A}^{\mathrm{hom}} + I)^{-1}f \\ &= \big(\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa u_\theta^\varepsilon - \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa c_\theta\big) + \big(\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa c_\theta - (\mathcal{A}^{\mathrm{hom}} + I)^{-1}f\big). \end{aligned} \tag{4.52}$$

For the first term $\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa u_\theta^\varepsilon - \mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa c_\theta$, we can use the Theorem 4.4.2, since $\mathcal{F}_\varepsilon$, $\mathcal{T}_\varepsilon$ and the multiplication by e_κ are unitary operators. The second term in (4.52) can be written as

$$
\begin{aligned}
&\mathcal{F}_\varepsilon^{-1}\mathcal{T}_\varepsilon^{-1}e_\kappa(\mathfrak{A}_\theta^{\rm hom}+I)^{-1}\widehat{f}(\theta,\varepsilon)-(2\pi)^{-3/2}\int_{\mathbb{R}^3}(\mathfrak{A}_\theta^{\rm hom}+I)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta d\theta\\
&=(2\pi)^{-3/2}\biggl(\int_{\varepsilon^{-1}Q'}(\mathfrak{A}_\theta^{\rm hom}+I)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta d\theta-\int_{\mathbb{R}^3}(\mathfrak{A}_\theta^{\rm hom}+I)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta d\theta\biggr)\\
&=-(2\pi)^{-3/2}\int_{\mathbb{R}^3\setminus\varepsilon^{-1}Q'}(\mathfrak{A}_\theta^{\rm hom}+I)^{-1}\widehat{f}(\theta,\varepsilon)e_\theta d\theta.
\end{aligned}
$$

It follows that there exists $C>0$ such that

$$
\begin{aligned}
\bigl\|u^\varepsilon-u^\varepsilon_{\rm hom}\bigr\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}&\le C\varepsilon\|f\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}+\frac{\varepsilon^2}{\|(A^{\rm hom})^{-1}\|^{-1}\pi^2+\varepsilon^2}\bigl\|\widehat{f}(\cdot,\varepsilon)\bigr\|_{L^2(\mathbb{R}^3)}\\
&=\biggl(C\varepsilon+\frac{\varepsilon^2}{\|(A^{\rm hom})^{-1}\|^{-1}\pi^2+\varepsilon^2}\biggr)\|f\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)},
\end{aligned}
$$

which implies the claim. □

We define

$$
N=\widetilde{N}+a_\theta,\quad a_\theta\in\mathbb{C}^{3\times3},\tag{4.53}
$$

where N solves (4.46) in the space $H^1_{\rm curl}(Q,d\mu)\cap\{u\mid\operatorname{div}u=0\}$, and the matrix a_θ is chosen so that

$$
\int_Q\theta\times A\bigl(\theta\times N(\theta\times c_\theta)\bigr)=0\quad\forall c_\theta\in\mathbb{C}^3.\tag{4.54}
$$

Note that a_θ is such that for all $\eta\in\Theta^\perp:=\{\eta\in\mathbb{C}^3,\quad\eta\cdot\theta=0\}$, $P_{\Theta^\perp}(a_\theta\eta)\neq0$, where with $P_{\Theta^\perp}$ is the orthogonal projection on $\Theta^\perp$.

We next show that there is at least a unique constant matrix $\widetilde{a}_\theta$ satisfying the condition (4.54).

Proposition 4.4.4. *There exists a unique $\widetilde{a}_\theta\in\mathbb{C}^{3\times3}$ such that*

$$
\widetilde{a}_\theta\theta=0,\qquad\widetilde{a}_\theta\eta\cdot\theta=0,\tag{4.55}
$$

and

$$
\int_Q\theta\times A\bigl(\theta\times\widetilde{a}_\theta\eta\bigr)=-\int_Q\theta\times A\bigl(\theta\times\widetilde{N}\eta\bigr)\qquad\forall\eta\in\Theta^\perp.\tag{4.56}
$$

Proof. For any orthogonal basis $\{e_1^\perp,e_2^\perp\}$ of $\Theta^\perp$, the identity (4.56) is equivalent to a linear system for the representation of the matrix $\widetilde{a}_\theta$ in the basis $\{\theta/|\theta|,e_1^\perp,e_2^\perp\}$. This system is uniquely solvable, subject to the conditions (4.55), for any right-hand side if and only if the only solution to the related homogeneous system is zero. The latter is easily verified, by noticing that if

$$\int_Q \theta \times A\big(\theta \times \widetilde{a}_\theta \eta\big) = 0 \qquad \forall \eta \in \Theta^\perp,$$

then, in particular,

$$\left(\int_Q A\right)\big(\theta \times \widetilde{a}_\theta \eta\big) \cdot (\theta \times a_\theta \eta) = 0,$$

from which we infer, due to the fact that A is positive definite, that $\theta \times \widetilde{a}_\theta \eta = 0$, and therefore $\widetilde{a}_\theta \eta = 0$ by the second condition in (4.55). Taking into account the first condition in (4.55), we obtain $\widetilde{a}_\theta = 0$, as required. □

Furthermore, we invoke the following statement.

Lemma 4.4.5. *One has*

$$\Theta^\perp = \{\theta \times c : c \in \mathbb{C}^3\}.$$

Proof. The inclusion $\{\theta \times c : c \in \mathbb{C}\} \subset \Theta^\perp$ is trivial. In order to show the opposite inclusion, we notice that for all $\eta \in \mathbb{C}^3$ there exists $\alpha \in \mathbb{C}^3$ such that

$$\theta \times (\theta \times \alpha) = \eta. \tag{4.57}$$

Indeed, the subspace of α such that $\theta \times (\theta \times \alpha) = 0$ consists of vectors parallel to θ, all of which are orthogonal to the right-hand side of (4.57). □

Using the above lemma, we write (4.56) in an equivalent form, as follows:

$$\int_Q \theta \times A\big(\theta \times a_\theta(\theta \times c)\big) = -\int_Q \theta \times A\big(\theta \times \widetilde{N}(\theta \times c)\big) \qquad \forall c \in \mathbb{C}^3, \tag{4.58}$$

which is the identity (4.54) we require.

Follows that we have an estimate uniform in θ for N defined in (4.53). Indeed we know from equation (4.46) that $\widetilde{N}$ is uniformly bounded in θ, and the same kind of estimate holds for a_θ using (4.58). Indeed we have that

$$\int_Q A|\theta \times a_\theta(\theta \times c_\theta)|^2 = -\int_Q A(\theta \times \widetilde{N}(\theta \times c_\theta)) \cdot (\theta \times a_\theta(\theta \times c_\theta))$$

Using the property of A and the conditions (4.55), we obtain

$$\|a_\theta(\theta \times c_\theta)\|_{L^2(Q,d\mu)} \leq \|\widetilde{N}(\theta \times c_\theta)\|_{L^2(Q,d\mu)}$$

which gives us the uniform estimate for a_θ.

In order to prove Theorem 4.4.2, we introduce the following decomposition for the vector function u_θ^ε, motivated by a formal asymptotic expansion in powers of ε.
For each $\varepsilon > 0$, $\theta \in \varepsilon^{-1}Q'$, we write

$$u_\theta^\varepsilon := U_\theta^\varepsilon + z_\theta^\varepsilon, \tag{4.59}$$

where

$$U_\theta^\varepsilon := c_\theta + i\varepsilon u_\theta^{(1)} + \varepsilon^2 R_\theta^\varepsilon, \qquad u_\theta^{(1)} := N(\theta \times c_\theta). \tag{4.60}$$

Here the second-order coefficient R_θ^ε is defined to be an element of $H^1_{\rm curl}(Q, d\mu)$ that solves

$$\begin{aligned}
&\overline{e_{\varepsilon\theta}}\,\mathrm{curl}\big(A\,\mathrm{curl}(e_{\varepsilon\theta}R_\theta^\varepsilon)\big) + \varepsilon^2\int_Q R_\theta^\varepsilon\, d\mu + \varepsilon^2\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}) \\
&= F - \varepsilon^{-2}\overline{e_{\varepsilon\theta}}\,\mathrm{curl}\big(A\,\mathrm{curl}(e_{\varepsilon\theta}c_\theta)\big) - i\varepsilon^{-1}\overline{e_{\varepsilon\theta}}\,\mathrm{curl}\big(A\,\mathrm{curl}\big(e_{\varepsilon\theta}u_\theta^{(1)}\big)\big) - c_\theta \\
&= F - i\theta \times A(i\theta \times c_\theta) - i\theta \times A\,\mathrm{curl}\big(N(i\theta \times c_\theta)\big) \\
&- i\,\mathrm{curl}\big(A\big(i\theta \times u_\theta^{(1)}\big)\big) + \varepsilon\theta \times A\big(i\theta \times u_\theta^{(1)}\big) - c_\theta := \mathcal{H}_\theta^\varepsilon,
\end{aligned} \tag{4.61}$$

where $\mathcal{H}_\theta^\varepsilon$ is treated as an element of the dual space $\big(H^1_{\rm curl}(Q, d\mu)\big)^*$, and the function $\Phi_{R_\theta^\varepsilon}$ is defined as in the decomposition (4.15). For all $\kappa \in Q'$ and $u \in H^1_{\rm curl}(Q, d\mu)$, we set

$$\mathrm{curl}(e_{\varepsilon\theta}u) = e_{\varepsilon\theta}(i\kappa \times u + \mathrm{curl}\, u). \tag{4.62}$$

Then, the second equality in (4.61) is verified by taking $\phi \in C^\infty_\#$ and noticing that

$$\begin{aligned}
&\Big\langle F - \varepsilon^{-2}\overline{e_{\varepsilon\theta}}\,\mathrm{curl}\big(A\,\mathrm{curl}(e_{\varepsilon\theta}c_\theta)\big) - i\varepsilon^{-1}\overline{e_{\varepsilon\theta}}\,\mathrm{curl}\big(A\,\mathrm{curl}\big(e_{\varepsilon\theta}u_\theta^{(1)}\big)\big) - c_\theta, \phi\Big\rangle \\
&= \int_Q F\cdot\overline{\phi} - \int_Q A\,\mathrm{curl}(N(i\theta \times c_\theta))\cdot\overline{(i\theta\times\phi)} - \int_Q A(i\theta\times c_\theta)\cdot\overline{(i\theta\times\phi)} \\
&- i\int_Q A\big(i\theta \times u_\theta^{(1)}\big)\cdot\overline{\mathrm{curl}\,\phi} - i\varepsilon\int_Q A\big(i\theta\times u_\theta^{(1)}\big)\cdot\overline{(i\theta\times\phi)} - c_\theta\cdot\overline{\int_Q\phi},
\end{aligned}$$

where we use (4.46).

Proposition 4.4.6. *For each $\varepsilon > 0$ and $\theta \in \varepsilon^{-1}Q'$ there exists a unique solution $R_\theta^\varepsilon \in H^1_{\rm curl}(Q, d\mu)$ for the problem* (4.61).

Proof. The problem (4.61) is understood as

$$\int_Q A\,\mathrm{curl}(e_\kappa R_\theta^\varepsilon)\cdot\overline{\mathrm{curl}(e_\kappa\varphi)} + \varepsilon^2\int_Q R_\theta^\varepsilon\overline{\int_Q\varphi} + \varepsilon^2\int_Q\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\cdot\overline{\varphi} = \langle\mathcal{H}_\theta^\varepsilon, \varphi\rangle$$
$$\forall\varphi \in H^1_{\rm curl}(Q, d\mu).$$

We apply the decomposition (4.15) to the function φ. By the Helmholtz decomposition, we have an orthogonality condition between $\overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta})$ and $\widetilde{\varphi} + \int_Q \varphi$. Thus the third term on the left-hand side of the above equation is

$$\begin{aligned}\int_Q \overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta}) \cdot \overline{\varphi} &= \int_Q \overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta}) \cdot \overline{\left(\widetilde{\varphi} + \int_Q \varphi + \overline{e_\kappa}\nabla(e_\kappa\Phi_\varphi)\right)} \\ &= \int_Q \nabla(e_\kappa\Phi_{R^\varepsilon_\theta}) \cdot \overline{\nabla(e_{\varepsilon\theta}\Phi_\varphi)}.\end{aligned}$$

Hence the proof is a result of Lax-Milgram theorem applied to the bilinear form

$$b(u,v) = \int_Q A\,\mathrm{curl}(e_\kappa u) \cdot \overline{\mathrm{curl}(e_\kappa v)} + \varepsilon^2 \int_Q u \cdot \overline{\int_Q v} + \varepsilon^2 \int_Q \nabla(e_\kappa\Phi_u) \cdot \overline{\nabla(e_\kappa\Phi_v)}$$

for all $u, v \in H^1_{\mathrm{curl}}(Q, d\mu)$ and Φ_u, Φ_v defined as in (4.13). Indeed, the form b is bounded and its coercivity is a consequence of the Poincaré-type inequality (4.18). $\square$

In order to prove the estimates for R^ε_θ in Theorem 4.4.7, we need to use the Poincaré-type inequality (4.18). Hence we would like to have the identity

$$\langle \mathcal{H}^\varepsilon_\theta, R^\varepsilon_\theta \rangle = \langle \mathcal{H}^\varepsilon_\theta, \widetilde{R}^\varepsilon_\theta \rangle$$

where $\widetilde{R}^\varepsilon_\theta$ is defined as in the decomposition (4.15). To prove it we need to analyse two properties of $\mathcal{H}^\varepsilon_\theta$. First of all, we observe that by the definition of $\mathcal{H}^\varepsilon_\theta$, see (4.61), one has

$$\left\langle \mathcal{H}^\varepsilon_\theta, \overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\phi) \right\rangle = 0 \quad \forall \phi \in H^1_\#,$$

since $\mathrm{div}(e_{\varepsilon\theta}c_\theta) = 0$ and $\mathrm{div}(e_{\varepsilon\theta}F) = 0$. In particular,

$$\left\langle \mathcal{H}^\varepsilon_\theta, \overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_u) \right\rangle = 0, \tag{4.63}$$

for all functions Φ_u that solve (4.13) for some $u \in H^1_\#(Q, d\mu)$. Furthermore, the functional $\mathcal{H}^\varepsilon_\theta$ vanishes on constant vector functions:

$$\langle \mathcal{H}^\varepsilon_\theta, d_\theta \rangle = 0 \quad \forall d_\theta \in \mathbb{C}^3. \tag{4.64}$$

This is a consequence of the equation (4.12) solved by c_θ, taking into account the condition (4.54).

4.4.1 Estimates for $\varepsilon^2 R_\theta^\varepsilon$

Theorem 4.4.7. *There exists $C>0$ such that for all $\varepsilon>0$, $\theta\in\varepsilon^{-1}Q'$, the solution R_θ^ε to the problem (4.61) satisfies the following estimates:*

$$\left\| R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon - \overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big) + \int_Q \overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big)\right\|_{L^2(Q,d\mu)} \le C\|F\|_{L^2(Q,d\mu)}, \tag{4.65}$$

$$\left\| \int_Q R_\theta^\varepsilon + \overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big) - \int_Q \overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big)\right\|_{L^2(Q,d\mu)} \le C\varepsilon^{-1}\|F\|_{L^2(Q,d\mu)}. \tag{4.66}$$

Proof. Suppose that $\phi_n\in C_\#^\infty$ converging to R_θ^ε in $L^2(Q,d\mu)$ are such that $\operatorname{curl}(e_{\varepsilon\theta}\phi_n)\to\operatorname{curl}(e_{\varepsilon\theta}R_\theta^\varepsilon)$ in $L^2(Q,d\mu)$ as $n\to\infty$, and use ϕ_n as test functions in the integral identity for (4.61):

$$\int_Q A\operatorname{curl}(e_{\varepsilon\theta}R_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_{\varepsilon\theta}\phi_n)} + \varepsilon^2\int_Q R_\theta^\varepsilon\cdot\overline{\int_Q\phi_n} + \varepsilon^2\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon})\cdot\overline{\phi_n} = \big\langle\mathcal{H}_\theta^\varepsilon,\phi_n\big\rangle. \tag{4.67}$$

Using the properties (4.63) and (4.64) for $\mathcal{H}_\theta^\varepsilon$, we write the right-hand side of the last equality as follows:

$$\big\langle\mathcal{H}_\theta^\varepsilon,\phi_n\big\rangle = \left\langle\mathcal{H}_\theta^\varepsilon,\phi_n-\int_Q R_\theta^\varepsilon - \overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big) + \int_Q\overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big)\right\rangle.$$

Furthermore, using the identity (*cf.* (4.62))

$$\begin{aligned}\operatorname{curl}&\left(\phi_n-\int_Q R_\theta^\varepsilon - \overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}) + \int_Q\overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big)\right) \\ &= \overline{e_{\varepsilon\theta}}\Bigg\{\operatorname{curl}\left(e_{\varepsilon\theta}\left(\phi_n-\int_Q R_\theta^\varepsilon - \overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}) + \int_Q\overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big)\right)\right) \\ &\quad-\nabla e_{\varepsilon\theta}\times\left(\phi_n-\int_Q R_\theta^\varepsilon - \overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}) + \int_Q\overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big)\right)\Bigg\},\end{aligned} \tag{4.68}$$

we rewrite (4.67) as

$$\begin{aligned}&\int_Q A\operatorname{curl}(e_{\varepsilon\theta}R_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_{\varepsilon\theta}\phi_n)} + \varepsilon^2\int_Q R_\theta^\varepsilon\cdot\overline{\int_Q\phi_n} + \varepsilon^2\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon})\cdot\overline{\phi_n} \\ &\quad= \int_Q\Big(F+\theta\times A(\theta\times c_\theta)+\theta\times A\big(\operatorname{curl}N(\theta\times c_\theta)\big)-c_\theta\Big) \\ &\quad\cdot\overline{\left(\phi_n-\int_Q R_\theta^\varepsilon - \overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}) + \int_Q\overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R_\theta^\varepsilon}\big)\right)}\end{aligned}$$

$$+\int_Q e_{\varepsilon\theta} A\big(\theta \times u_\theta^{(1)}\big) \cdot \overline{\operatorname{curl}\left(e_{\varepsilon\theta}\left(\phi_n - \int_Q R_\theta^\varepsilon - \overline{e_{\varepsilon\theta}} \nabla\big(e_{\varepsilon\theta} \Phi_{R_\theta^\varepsilon}\big) + \int_Q \overline{e_{\varepsilon\theta}} \nabla\big(e_{\varepsilon\theta} \Phi_{R_\theta^\varepsilon}\big)\right)\right)}$$

In the last identity we pass to the limit as $n \to \infty$.

Applying the decomposition (4.15) to the function R_θ^ε, due to the property (4.17), the second term on the left-hand side of the resulting equality is

$$\int_Q R_\theta^\varepsilon \cdot \overline{\int_Q R_\theta^\varepsilon} = \left| \int R_\theta^\varepsilon \right|^2.$$

Due to the orthogonality between $\overline{e_\kappa} \nabla(e_\kappa \Phi_{R_\theta^\varepsilon})$ and $\widetilde{R}_\varepsilon^\theta + \int R_\theta^\varepsilon$ established by the Helmholtz decomposition, the third term on the left-hand side is

$$\int_Q \overline{e_{\varepsilon\theta}} \nabla(e_{\varepsilon\theta} \Phi_{R_\theta^\varepsilon}) \cdot \overline{R_\theta^\varepsilon} = \int \big|\nabla(e_{\varepsilon\theta} \Phi_{R_\theta^\varepsilon})\big|^2.$$

Hence, we obtain

$$\int_Q A \operatorname{curl}(e_{\varepsilon\theta} R_\theta^\varepsilon) \cdot \overline{\operatorname{curl}(e_{\varepsilon\theta} R_\theta^\varepsilon)} + \varepsilon^2 \left| \int R_\theta^\varepsilon \right|^2 + \varepsilon^2 \int \big|\nabla(e_{\varepsilon\theta} \Phi_{R_\theta^\varepsilon})\big|^2 \tag{4.69}$$

$$= \int_Q \Big(F + \theta \times A(\theta \times c_\theta) + \theta \times A\big(\operatorname{curl} N(\theta \times c_\theta)\big) - c_\theta\Big) \cdot \overline{\left(\widetilde{R_\theta^\varepsilon} + \int_Q \overline{e_{\varepsilon\theta}} \nabla\big(e_{\varepsilon\theta} \Phi_{R_\theta^\varepsilon}\big)\right)}$$

$$+ \int_Q e_{\varepsilon\theta} A\big(\theta \times u_\theta^{(1)}\big) \cdot \overline{\operatorname{curl}\left(e_{\varepsilon\theta}\left(\widetilde{R_\theta^\varepsilon} + \int_Q \overline{e_{\varepsilon\theta}} \nabla\big(e_{\varepsilon\theta} \Phi_{R_\theta^\varepsilon}\big)\right)\right)},$$

where the last term can be replaced by

$$\int_Q e_{\varepsilon\theta} A\big(\theta \times u_\theta^{(1)}\big) \cdot \overline{\operatorname{curl}\big(e_{\varepsilon\theta} \widetilde{R_\theta^\varepsilon}\big)},$$

due to condition (4.54).

Next, consider $\xi_\theta^\varepsilon \in H^1_{\mathrm{curl}}(Q, d\mu)$ that solves

$$\overline{e_{\varepsilon\theta}} \operatorname{curl}\big(A \operatorname{curl}(e_{\varepsilon\theta} \xi_\theta^\varepsilon)\big) + \varepsilon^2 \int_Q \xi_\theta^\varepsilon + \varepsilon^2 \overline{e_{\varepsilon\theta}} \nabla(e_{\varepsilon\theta} \Phi_{\xi_\theta^\varepsilon}) = \overline{e_{\varepsilon\theta}} \operatorname{curl}\big(e_{\varepsilon\theta} A\big(\theta \times u_\theta^{(1)}\big)\big). \tag{4.70}$$

The existence and the uniqueness of solution $\xi_\theta^\varepsilon \in H^1_{\mathrm{curl}}(Q, d\mu)$ are a consequence of the same argument used in Proposition 4.4.6. Furthermore, considering (4.70) with ξ_θ^ε as test function, we have the uniform estimate

$$\big\|\operatorname{curl}(e_{\varepsilon\theta} \xi_\theta^\varepsilon)\big\|_{L^2(Q,d\mu)} \le C \|F\|_{L^2(Q,d\mu)}. \tag{4.71}$$

Next, testing (4.70) with

$$\widetilde{R^\varepsilon_\theta} = R^\varepsilon_\theta - \int_Q R^\varepsilon_\theta - \overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta}),$$

we write the last term in the right hand side of (4.69), as

$$\begin{aligned}\int_Q e_{\varepsilon\theta}A\big(\theta\times u^{(1)}_\theta\big)\cdot\overline{\operatorname{curl}\big(e_{\varepsilon\theta}\widetilde{R^\varepsilon_\theta}\big)} &= \int_Q A\operatorname{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)\cdot\overline{\operatorname{curl}\big(e_{\varepsilon\theta}\widetilde{R^\varepsilon_\theta}\big)}\\ &\quad+\varepsilon^2\int_Q\xi^\varepsilon_\theta\cdot\overline{\int_Q\widetilde{R^\varepsilon_\theta}}+\varepsilon^2\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})\cdot\overline{\widetilde{R^\varepsilon_\theta}}.\end{aligned}$$

At the same time, we have

$$\int_Q\xi^\varepsilon_\theta\cdot\overline{\int_Q\widetilde{R^\varepsilon_\theta}}=\int_Q\xi^\varepsilon_\theta\cdot\overline{\int_Q\Big(R^\varepsilon_\theta-\int_Q R^\varepsilon_\theta-\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\Big)}=-\int_Q\xi^\varepsilon_\theta\cdot\overline{\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})},$$

and, by equation (4.13) yields

$$\begin{aligned}\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})\cdot\overline{\widetilde{R^\varepsilon_\theta}}&=\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})\cdot\overline{\Big(R^\varepsilon_\theta-\int_Q R^\varepsilon_\theta-\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\Big)}\\&=-\int\overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta}\big)\cdot\overline{\int_Q R^\varepsilon_\theta}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_Q e_{\varepsilon\theta}A\big(\theta\times u^{(1)}_\theta\big)\cdot\overline{\operatorname{curl}\big(e_{\varepsilon\theta}\widetilde{R^\varepsilon_\theta}\big)} &= \int_Q A\operatorname{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)\cdot\overline{\operatorname{curl}\big(e_{\varepsilon\theta}\widetilde{R^\varepsilon_\theta}\big)}\\ &\quad-\varepsilon^2\int_Q\xi^\varepsilon_\theta\cdot\overline{\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})}-\varepsilon^2\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})\cdot\overline{\int_Q R^\varepsilon_\theta}.\end{aligned}\tag{4.72}$$

We would like to rewrite the expression on the right-hand side of (4.72) using ξ^ε_θ as a test function in the integral identity (4.61). Notice first that, for a general measure μ, the curl of an arbitrary function in $H^1_{\rm curl}(Q,d\mu)$ is not uniquely defined. However for the solution ξ^ε_θ to (4.70) there exists a natural choice of the curl ξ^ε_θ. Indeed, consider sequences ϕ_n, $\psi_n\in C^\infty_\#$ converging to ξ^ε_θ in $L^2(Q,d\mu)$, so that

$$\operatorname{curl}(e_{\varepsilon\theta}\phi_n)\to\operatorname{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)\qquad\operatorname{curl}(e_{\varepsilon\theta}\psi_n)\to\operatorname{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta).$$

The difference $\operatorname{curl}(e_{\varepsilon\theta}\phi_n)-\operatorname{curl}(e_{\varepsilon\theta}\psi_n)$ converges to zero, and hence so does $\operatorname{curl}\phi_n-\operatorname{curl}\psi_n$. Henceforth we denote by $\operatorname{curl}\xi^\varepsilon_\theta$ the common L^2-limit of curl $\operatorname{curl}\phi_n$ for sequences $\phi_n\in C^\infty_\#$ with the above properties.

The unique choice of the curl ξ^ε_θ as above allows us to write

$$\int_Q A\,\mathrm{curl}(e_{\varepsilon\theta}R^\varepsilon_\theta)\cdot\overline{\mathrm{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)}+\varepsilon^2\int_Q R^\varepsilon_\theta\cdot\overline{\int_Q\xi^\varepsilon_\theta}+\varepsilon^2\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\cdot\overline{\xi^\varepsilon_\theta}=\big\langle\mathcal{H}^\varepsilon_\theta,\xi^\varepsilon_\theta\big\rangle.$$

Applying the decomposition (4.15) to ξ^ε_θ and using the property (4.16), we have

$$\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\cdot\overline{\xi^\varepsilon_\theta}=\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\cdot\overline{\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})}=\int_Q\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\cdot\overline{\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})}.$$

Recalling also the properties (4.63) and (4.64) of $\mathcal{H}^\varepsilon_\theta$, we obtain

$$\begin{aligned}\int_Q A\,\mathrm{curl}(e_{\varepsilon\theta}R^\varepsilon_\theta)\cdot\overline{\mathrm{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)}+\varepsilon^2\int_Q R^\varepsilon_\theta\cdot\overline{\int_Q\xi^\varepsilon_\theta}+\varepsilon^2\int_Q\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\cdot\overline{\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})}\\=\big\langle\mathcal{H}^\varepsilon_\theta,\xi^\varepsilon_\theta\big\rangle=\big\langle\mathcal{H}^\varepsilon_\theta,\widetilde{\xi^\varepsilon_\theta}\big\rangle,\end{aligned}$$

and therefore

$$\int_Q A\,\mathrm{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)\cdot\overline{\mathrm{curl}(e_{\varepsilon\theta}R^\varepsilon_\theta)}+\varepsilon^2\int_Q\xi^\varepsilon_\theta\cdot\overline{\int_Q R^\varepsilon_\theta}+\varepsilon^2\int_Q\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})\cdot\overline{\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})}=\overline{\big\langle\mathcal{H}^\varepsilon_\theta,\widetilde{\xi^\varepsilon_\theta}\big\rangle}.\tag{4.73}$$

We now rewrite the equation (4.72) using (4.73), as follows:

$$\begin{aligned}\int_Q e_{\varepsilon\theta}A\big(\theta\times u^{(1)}_\theta\big)\cdot\overline{\mathrm{curl}\,\big(e_{\varepsilon\theta}\widetilde{R^\varepsilon_\theta}\big)}&=\overline{\big\langle\mathcal{H}^\varepsilon_\theta,\widetilde{\xi^\varepsilon_\theta}\big\rangle}\qquad(4.74)\\&-\overline{\int_Q R^\varepsilon_\theta}\cdot\bigg(\int_Q A\,\mathrm{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)\cdot\overline{\mathrm{curl}\,e_{\varepsilon\theta}}+\varepsilon^2\int_Q\xi^\varepsilon_\theta+\varepsilon^2\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})\bigg)\\&-\varepsilon^2\bigg(\int_Q\xi^\varepsilon_\theta\cdot\overline{\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})}+\int_Q\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{\xi^\varepsilon_\theta})\cdot\overline{\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})}\bigg).\end{aligned}$$

The second term on the right-hand side of (4.74) vanishes, by using the unity as a test function in the integral identity for (4.70) and noting that

$$\int e_{\varepsilon\theta}A\big(\theta\times u^{(1)}_\theta\big)\cdot\overline{\mathrm{curl}\,e_{\varepsilon\theta}}=i\varepsilon\int_Q\theta\times A\big(\theta\times u^{(1)}_\theta\big)=0,$$

in view of (4.54). The third term on the right-hand side of (4.74) also vanishes, by using $\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})$ as a test function in the integral formulation for (4.70) and taking advantage of the fact that curl vanishes on gradient fields.

Returning to (4.69), we thus obtain

$$\int_Q A\,\mathrm{curl}(e_{\varepsilon\theta}R^\varepsilon_\theta)\cdot\overline{\mathrm{curl}(e_{\varepsilon\theta}R^\varepsilon_\theta)}+\varepsilon^2\left|\int R^\varepsilon_\theta\right|^2+\varepsilon^2\int\left|\nabla(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta})\right|^2$$

$$=\int_Q\Big(F+\theta\times A(\theta\times c_\theta)+\theta\times A\big(\mathrm{curl}\,N(\theta\times c_\theta)\big)-c_\theta\Big)\cdot\overline{\left(\widetilde{R^\varepsilon_\theta}+\int_Q\overline{e_{\varepsilon\theta}}\nabla\big(e_{\varepsilon\theta}\Phi_{R^\varepsilon_\theta}\big)\right)}$$

$$+\overline{\big\langle\mathcal{H}^\varepsilon_\theta,\widetilde{\xi^\varepsilon_\theta}\big\rangle}. \tag{4.75}$$

Lemma 4.4.8. *The last term on the right hand side of* (4.75) *is bounded uniformly in* ε *and* θ*:*

$$\left|\big\langle\mathcal{H}^\varepsilon_\theta,\widetilde{\xi^\varepsilon_\theta}\big\rangle\right|\le C\|F\|_{L^2(Q,d\mu)},\quad C>0.$$

Proof. It follows by the definition of $\mathcal{H}^\varepsilon_\theta$, see (4.61), that

$$\big\langle\mathcal{H}^\varepsilon_\theta,\widetilde{\xi^\varepsilon_\theta}\big\rangle=\int_Q\Big(F+\theta\times A(\theta\times c_\theta)+\theta\times A\big(\mathrm{curl}\,N(\theta\times c_\theta)\big)-c_\theta+i\varepsilon\theta\times A\big(\theta\times u^{(1)}_\theta\big)\Big)\cdot\overline{\widetilde{\xi^\varepsilon_\theta}}$$

$$+\int_Q A(\theta\times u^{(1)}_\theta)\cdot\overline{\mathrm{curl}\,\widetilde{\xi^\varepsilon_\theta}}.$$

Recalling the formula (4.62), we write (*cf.* (4.68))

$$\overline{\mathrm{curl}\,\widetilde{\xi^\varepsilon_\theta}}=e_{\varepsilon\theta}\overline{\mathrm{curl}\big(e_{\varepsilon\theta}\widetilde{\xi^\varepsilon_\theta}\big)}+i\varepsilon\theta\times\overline{\widetilde{\xi^\varepsilon_\theta}},$$

and thus

$$\big\langle\mathcal{H}^\varepsilon_\theta,\widetilde{\xi^\varepsilon_\theta}\big\rangle=\int_Q\big(F+\theta\times A(\theta\times c_\theta)+\theta\times A\big(\mathrm{curl}\,N(\theta\times c_\theta)\big)-c_\theta\big)\cdot\overline{\widetilde{\xi^\varepsilon_\theta}}+\int_Q e_{\varepsilon\theta}A\big(\theta\times u^{(1)}_\theta\big)\cdot\overline{\mathrm{curl}(e_{\varepsilon\theta}\xi^\varepsilon_\theta)}, \tag{4.76}$$

since

$$\int_Q e_{\varepsilon\theta}A\big(\theta\times u^{(1)}_\theta\big)\cdot\overline{\mathrm{curl}\left(e_{\varepsilon\theta}\int_Q\xi^\varepsilon_\theta\right)}=i\varepsilon\int_Q\theta\times A\big(\theta\times u^{(1)}_\theta\big)\cdot\overline{\int_Q\xi^\varepsilon_\theta}=0,$$

by the condition (4.54).

Applying the Hölder inequality to the right-hand side of the equation (4.76), using the Poincaré inequality (4.18) for ξ^ε_θ, and taking into account the estimate (4.71) yields the required statement. $\square$

Combining the Lemma 4.4.8, the Poincaré inequality (4.18) for R^ε_θ and the Hölder inequality for the first term on the right-hand side of the equation (4.75), we obtain the uniform bound

$$\left\|\operatorname{curl}(e_{\varepsilon\theta}R_\theta^\varepsilon)\right\|_{L^2(Q,d\mu)} \leq C\|F\|_{L^2(Q,d\mu)}. \tag{4.77}$$

Finally, the estimate (4.77) combined with (4.18) for $u = R_\theta^\varepsilon$ implies the estimate (4.65). The same bound, Lemma 4.4.8, and the equation (4.75) imply the estimate (4.66). □

Corollary 4.4.9. *There exists $C > 0$ such that the following estimate holds uniformly in ε, θ and F :*

$$\|U_\theta^\varepsilon - c_\theta\|_{L^2(Q,d\mu)} \leq C\varepsilon\|F\|_{L^2(Q,d\mu)}.$$

4.4.2 Conclusion of the convergence estimate

Proposition 4.4.10. *There exists $C > 0$ such that the function z_θ^ε in* (4.59) *satisfies the estimate*

$$\|z_\theta^\varepsilon\|_{L^2(Q,d\mu)} \leq C\varepsilon\|F\|_{L^2(Q,d\mu)}, \quad \varepsilon > 0,\ \theta \in \varepsilon^{-1}Q',\ F \in L^2(Q). \tag{4.78}$$

Proof. The function $z_\theta^\varepsilon \in H^1_{\operatorname{curl}}(Q, d\mu)$ solves the problem

$$\varepsilon^{-2}\overline{e_{\varepsilon\theta}}\operatorname{curl} A \operatorname{curl}(e_{\varepsilon\theta}z_\theta^\varepsilon) + z_\theta^\varepsilon = -i\varepsilon u_\theta^{(1)} - \varepsilon^2\widetilde{R_\theta^\varepsilon}. \tag{4.79}$$

Using z_θ^ε as a test function in the integral formulation of (4.79), we obtain

$$\varepsilon^{-2}\int_Q A\operatorname{curl}(e_{\varepsilon\theta}z_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_{\varepsilon\theta}z_\theta^\varepsilon)} + \int_Q |z_\theta^\varepsilon|^2 = -i\varepsilon\int_Q u_\theta^{(1)}\cdot\overline{z_\theta^\varepsilon} - \varepsilon^2\int_Q \widetilde{R_\theta^\varepsilon}\cdot\overline{z_\theta^\varepsilon}. \tag{4.80}$$

Using the estimate

$$\|R_\theta^\varepsilon\|_{L^2(Q,d\mu)} \leq C\varepsilon^{-1}\|F\|_{L^2(Q,d\mu)},$$

which follows from (4.65) and (4.66), the elliptic estimate for the equation

$$\overline{e_\kappa}\triangle(e_\kappa\Phi_{R_\theta^\varepsilon}) = \overline{e_\kappa}\operatorname{div}(e_\kappa R_\theta^\varepsilon),$$

and then the observation

$$\widetilde{R_\theta^\varepsilon} = \left\{R_\theta^\varepsilon - \int_Q R_\theta^\varepsilon - \overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\right\} + \int_Q \overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon}),$$

we infer from (4.65) that

$$\|\widetilde{R_\theta^\varepsilon}\|_{L^2(Q,d\mu)} \leq C\varepsilon^{-1}\|F\|_{L^2(Q,d\mu)}. \tag{4.81}$$

Now, by using the Hölder inequality for the right-hand side of (4.80), then the formula

(4.49) and the estimate (4.81), we obtain (4.78). $\square$

Note that by using Hölder inequality and (4.78) in the right hand side of (4.80), we can obtain

$$\| \operatorname{curl}(e_{\varepsilon\theta} z_\theta^\varepsilon)\|_{L^2(Q,d\mu)} \le C\varepsilon^2 \|F\|_{L^2(Q,d\mu)}, \quad \varepsilon > 0,\ \theta \in \varepsilon^{-1}Q',\ F \in L^2(Q). \tag{4.82}$$

Combining Corollary 4.4.9 and Proposition 4.4.10, we obtain (4.48), since

$$\|u_\theta^\varepsilon - c_\theta\|_{L^2(Q,d\mu)} \le \|z_\theta^\varepsilon\|_{L^2(Q,d\mu)} + \|U_\theta^\varepsilon - c_\theta\|_{L^2(Q,d\mu)},$$

which concludes the proof of Theorem 4.4.2.

4.5 Estimates for electric field and electric displacement

As discussed in the Introduction, Theorem 4.4.2 concerns the Maxwell system in the non-magnetic case and without external currents, written in terms of the magnetic field. Hence, the estimate (4.51) holds for magnetic field H_ε and magnetic induction B_ε (which coincide in this setting). We complete the analysis by establishing estimates for the electric field E_ε and the electric displacement

$$D_\varepsilon = A(\cdot/\varepsilon)^{-1}E_\varepsilon, \tag{4.83}$$

where the matrix A is the inverse of the relative dielectric permittivity. In order to obtain these estimates, we write the main equation (4.1) in terms of the Maxwell system

$$\begin{cases} \operatorname{curl} A(\cdot/\varepsilon)D_\varepsilon + H_\varepsilon = f^\varepsilon, \\ \operatorname{curl} H_\varepsilon = D_\varepsilon, \end{cases} \tag{4.84}$$

where D_ε, H_ε and f^ε are divergence free. The homogenised problem for (4.84), is given by (4.3), which corresponds to

$$\begin{cases} \operatorname{curl} A^{\mathrm{hom}} D_\varepsilon^{\mathrm{hom}} + H_\varepsilon^{\mathrm{hom}} = f, \\ \operatorname{curl} H_\varepsilon^{\mathrm{hom}} = D_\varepsilon^{\mathrm{hom}}, \end{cases} \tag{4.85}$$

where A^{hom} is defined in (4.50), and the homogenised asymptotic values of the electric displacement and electric field are linked by the formula $D_\varepsilon^{\mathrm{hom}} = (A^{\mathrm{hom}})^{-1}E_\varepsilon^{\mathrm{hom}}$.

As in Section 4.1.1, starting from (4.84) one obtains the following transformed system

$$\begin{cases} \varepsilon^{-1}\overline{e_\kappa} \operatorname{curl} Ae_\kappa D_\theta^\varepsilon + H_\theta^\varepsilon = F, \\ \varepsilon^{-1}\overline{e_\kappa} \operatorname{curl} e_\kappa H_\theta^\varepsilon = D_\theta^\varepsilon, \end{cases} \tag{4.86}$$

where H_θ^ε coincides with u_θ^ε defined in (4.11), and $D_\theta^\varepsilon := \overline{e_\kappa}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon D_\varepsilon$. Recall that F is $\operatorname{div} e_\kappa$-free, and so are the fields D_θ^ε, H_θ^ε. Regarding the transformed electric field E_θ^ε, follows from (4.83) that

$$E_\theta^\varepsilon = AD_\theta^\varepsilon. \tag{4.87}$$

To find the right approximation for D_θ^ε, we use the one developed for H_θ^ε. Substituting (4.59) into the second line of (4.86), one has

$$\begin{aligned} D_\theta^\varepsilon &= \varepsilon^{-1}\overline{e_\kappa}\operatorname{curl} e_\kappa(c_\theta + \varepsilon N(i\theta\times c_\theta) + \varepsilon^2 R_\theta^\varepsilon + z_\theta^\varepsilon) \\ &= (\operatorname{curl} N + I)(i\theta\times c_\theta) + \varepsilon\big(i\theta\times N(i\theta\times c_\theta) + \overline{e_\kappa}\operatorname{curl}(e_\kappa R_\theta^\varepsilon)\big) + \varepsilon^{-1}\overline{e_\kappa}\operatorname{curl}(e_\kappa z_\theta^\varepsilon), \end{aligned}$$

where c_θ solves (4.49), N is defined in (4.53) and R_θ^ε is the solution of (4.61). As a consequence of (4.77) and (4.82) we can obtain the following result.

Theorem 4.5.1. *Under the assumptions on the measure μ and the coefficient A stated in Theorem 4.4.2, there exists a constant $C > 0$ independent of θ, ε and F such that, for D_θ^ε solving* (4.86) *and for E_θ^ε defined in* (4.87)*, hold the following estimates*

$$\|D_\theta^\varepsilon - (\operatorname{curl} N + I)(i\theta\times c_\theta)\|_{L^2(Q,d\mu)} \le \varepsilon C\|F\|_{L^2(Q,d\mu)}, \tag{4.88}$$

$$\|E_\theta^\varepsilon - A(\operatorname{curl} N + I)(i\theta\times c_\theta)\|_{L^2(Q,d\mu)} \le \varepsilon C\|F\|_{L^2(Q,d\mu)}. \tag{4.89}$$

As done with Corollary 4.4.3, it is possible to prove the following estimates in the whole space

Corollary 4.5.2. *There exists a constant $C > 0$ independent of ε and f such that*

$$\|D_\varepsilon - (\operatorname{curl} N(\cdot/\varepsilon) + I)D_\varepsilon^{\mathrm{hom}}\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \le \varepsilon C\|f\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}, \tag{4.90}$$

$$\|E_\varepsilon - A(\cdot/\varepsilon)(\operatorname{curl} N(\cdot/\varepsilon) + I)(A^{\mathrm{hom}})^{-1}E_\varepsilon^{\mathrm{hom}}\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \le \varepsilon C\|f\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}, \tag{4.91}$$

where D_ε solves (4.84)*, E_ε is defined in* (4.83)*. $D_\varepsilon^{\mathrm{hom}}$ is a solution of the homogenised problem* (4.85)*, and $E_\varepsilon^{\mathrm{hom}} = A^{\mathrm{hom}}D_\varepsilon^{\mathrm{hom}}$.*

Let us note that contrary to the estimate (4.51) for the magnetic field and induction, the estimates (4.90) and (4.91) for the electric displacement and field contain terms rapidly oscillating as $\varepsilon \to 0$. Indeed, there is a "zero-order corrector" in the leading order term of approximation for D_θ^ε (and consequently for E_θ^ε). This role is played by the matrices $\{\operatorname{curl} N\}$ and $\{A(\operatorname{curl} N + I)(A^{\mathrm{hom}})^{-1} - I\}$ which are ε-dependent terms in the estimates. Notice that the two matrices have zero mean, thus the classical result of weak convergence to zero for $D_\varepsilon - D_\varepsilon^{\mathrm{hom}}$ and $E_\varepsilon - E_\varepsilon^{\mathrm{hom}}$ is valid. In order to have norm resolvent estimates, we need to add an oscillating element in the first term of the approximation. Similar observation were made in [10] for the case when μ is the Lebesgue measure.

Chapter 5

Operator-norm homogenisation estimates for Maxwell equations on periodic non-magnetic singular structures: the case of non-zero current

Introduction

The aim of this chapter is a natural continuation of the results obtained in Chapter 4, namely norm-resolvent homogenisation estimates for the stationary Maxwell system of electromagnetism with non-zero external current, in the setting of arbitrary periodic (Borel) measures. This result can be found in the first part of the preprint [21] by Cherednichenko and D'Onofrio. In Chapter 4 (see also [20]) we proved operator-norm resolvent estimates for the Maxwell system in absence of external currents and with relative magnetic permeability set to unity. The related problem took the form (*c.f.* (4.1))

$$\operatorname{curl} A(\cdot/\varepsilon) \operatorname{curl} u^\varepsilon + u^\varepsilon = f^\varepsilon, \quad f^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \operatorname{div} f^\varepsilon = 0 \quad \varepsilon > 0, \tag{5.1}$$

where, for each ε, the ε-periodic measure μ^ε is given by rescaling a fixed Q-periodic Borel measure $Q = [0,1)^3$ and A is a symmetric, bounded and Q-periodic uniformly positive matrix-valued function. Here u^ε represents the divergence-free magnetic field and the matrix A stands for the inverse of the relative dielectric permittivity (see Introduction of Chapter 3 for more details).

The approach we developed in Chapter 4 to obtain norm-resolvent convergence estimates as $\varepsilon \to 0$, is based on the study of a family of operators, parametrised by the quasimomentum θ, obtained from (5.1) by the Floquet transform (see Section 4.1.1). This related strategy consists of the following steps: the construction of an asymptotic approximation in powers of ε, the analysis of the homogenisation corrector as a function of ε and θ, and a convergence estimate uniform with respect to θ as $\varepsilon \to 0$ for the remainder. One of our principal tools is a special Poincaré-type inequality in Sobolev spaces of quasiperiodic functions, which takes into account the fact that we are dealing with arbitrary measures. The results of Chapter 4 allow us to approximate the magnetic field and the magnetic induction directly with the solution of the related homogenised equation. Different estimates are obtained for the electric field and electric displacement, where the approximation contains rapidly oscillating terms of zero order.

In the present chapter we adapt the method described above to the system of Maxwell equations with unitary relative magnetic permeability and non-zero external current, again in the case of an arbitrary periodic Borel measure μ. In the particular case when μ is the Lebesgue measure, this problem has been analysed by Birman and Suslina in [10], where norm resolvent estimates were obtained with a method based on the analysis of a "spectral germ", which originated in [8], [9]. The spectral analysis of [10] allows the construction of a special corrector which depends on ε and enters the leading-order term of the approximation. Our approach is based on a new asymptotic expansion, which is closer in spirit to the classical power-series approach and allows us to obtain estimates in the setting of singular periodic structures. As opposed to the approach of [10], in our analysis the corrector is captured by a suitable Poincaré inequality and is included directly in the leading order term of the approximation.

As in Introduction of Chapter 4, we consider a Q-periodic Borel measure μ in $\mathbb{R}^3$, where $Q = [0,1)^3$, such that $\mu(Q) = 1$. For each $\varepsilon > 0$ the ε-periodic measure μ^ε is defined as $\mu^\varepsilon(B) = \varepsilon^3 \mu(\varepsilon^{-1} B)$ for every Borel set $B \subset \mathbb{R}^3$.

We analyse the asymptotic behaviour, as $\varepsilon \to 0$, of solutions to the system of Maxwell equations with relative magnetic permeability set to unity and non-zero external current:

$$\begin{cases} \operatorname{curl}(A(\cdot/\varepsilon)\mathcal{D}_\varepsilon) + \mathcal{B}_\varepsilon = 0, \\ \operatorname{curl} \mathcal{B}_\varepsilon - \mathcal{D}_\varepsilon = g^\varepsilon, \end{cases} \tag{5.2}$$

where $g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ is a divergence-free field representing the current density. In the equation (5.2) $\mathcal{B}_\varepsilon$ is the magnetic induction, $\mathcal{D}_\varepsilon$ is the electric displacement, $\mathcal{H}_\varepsilon$ is the magnetic field (which coincides with $\mathcal{B}_\varepsilon$ throughout this chapter), and $\mathcal{E}_\varepsilon = A\mathcal{D}_\varepsilon$ is the electric field. Note that we have slightly changed the notation with respect the Maxwell system (3.11) introduced in Chapter 3 for convenience of the presentation to follow. Here A, which stands for the inverse of the relative dielectric permittivity, is a real-valued continuously differentiable Q-periodic matrix-valued function, which is assumed to be symmetric and positive definite.

Our goal here is to obtain norm-resolvent estimates for the difference between the solution of (5.2) for small ε and the solution to a suitable homogenised problem which serves as a replacement to the formally suggested system

$$\begin{cases} \operatorname{curl}(A^{\text{hom}}\mathcal{D}^{\text{hom}}_\varepsilon) + \mathcal{B}^{\text{hom}}_\varepsilon = 0, \\ \operatorname{curl}\mathcal{B}^{\text{hom}}_\varepsilon - \mathcal{D}^{\text{hom}}_\varepsilon = g^\varepsilon, \end{cases} \tag{5.3}$$

where $g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, $\operatorname{div} g^\varepsilon = 0$, and A^{hom} is the matrix of "effective" homogenised material coefficients. As discussed at the end of Chapter 3, the standard formal two-scale asymptotic result in the "limit" system (5.3), which turns out to be an incorrect model if one were to require norm-resolvent (or even strong) convergence as $\varepsilon \to 0$. We will demonstrate that the correct replacement of (5.3) involves an ε-dependent pseudodifferential operator, which is in some sense, a singular perturbation of (5.3).

Our first step in tackling the system (5.2) is to rewrite it in a symmetric form, for which we follow the approach of [58, Section 3]. Labelling $A^{1/2}\mathcal{D}_\varepsilon := D_\varepsilon$, we have that (5.2) is equivalent to

$$A^{1/2}\operatorname{curl}\operatorname{curl}(A^{1/2}D_\varepsilon) + D_\varepsilon = -A^{1/2}g^\varepsilon, \quad g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \operatorname{div} g^\varepsilon = 0. \tag{5.4}$$

We denote by $C^1_0(\mathbb{R}^3)$ the set of complex-valued differentiable vector functions, with compact support in $\mathbb{R}^3$. We define the space $H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$ as the closure of the set of pairs

$$\{(\phi, \operatorname{curl} A^{1/2}\phi) : \ \phi \in C^1_0(\mathbb{R}^3)\}$$

in the direct sum $L^2(\mathbb{R}^3, d\mu^\varepsilon) \oplus L^2(\mathbb{R}^3, d\mu^\varepsilon)$. We say that

$$(D_\varepsilon, \operatorname{curl}(A^{1/2}D_\varepsilon)) \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$$

is the solution of (5.4) if

$$\int_{\mathbb{R}^3} \operatorname{curl}(A^{1/2}D_\varepsilon) \cdot \overline{\operatorname{curl}(A^{1/2}\phi)} + \int_{\mathbb{R}^3} D_\varepsilon \cdot \overline{\phi} = -\int_{\mathbb{R}^3} A^{1/2}g^\varepsilon \cdot \overline{\phi} \qquad \forall (\phi, \operatorname{curl} A^{1/2}\phi) \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon). \tag{5.5}$$

For every $\varepsilon > 0$ the left-hand side of (5.5) defines an inner product in $H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$. The right-hand side is a linear bounded functional on $H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$, hence the existence and uniqueness of the solution of (5.5) are a consequence of the Riesz representation theorem.

In what follows we study the resolvent of the operator $\mathcal{A}^\varepsilon$ with domain

$$\begin{aligned} \operatorname{dom}(\mathcal{A}^\varepsilon) = \{ & u \in L^2(\mathbb{R}^3, d\mu^\varepsilon) : \ \exists \operatorname{curl} A^{1/2}u \text{ such that} \\ & \int_{\mathbb{R}^3} \operatorname{curl}(A^{1/2}u) \cdot \overline{\operatorname{curl}(A^{1/2}\phi)} + \int_{\mathbb{R}^3} u \cdot \overline{\phi} = -\int_{\mathbb{R}^3} A^{1/2}g \cdot \overline{\phi} \quad \forall \phi \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon), \\ & \text{for some } g \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \ \operatorname{div} g = 0\}, \end{aligned} \tag{5.6}$$

defined by the formula $\mathcal{A}^\varepsilon u = -A^{1/2}g - u$, where $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, $\operatorname{div} g = 0$, and $u \in \operatorname{dom}(\mathcal{A}^\varepsilon)$ are linked as in (5.6). Note that in general, for a given function $u \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ there exists more than one element $(u, \operatorname{curl} A^{1/2}u) \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$. However, for each $u \in \operatorname{dom}(\mathcal{A}^\varepsilon)$ there exists only one $\operatorname{curl}(A^{1/2}u)$ such that (5.6) holds, so the solution to (5.6) is uniquely defined as a pair $(u, \operatorname{curl} A^{1/2}u)$. Similarly to Chapter 4, we notice that $\operatorname{dom}(\mathcal{A}^\varepsilon)$ is dense in $L^2(\mathbb{R}^3, d\mu^\varepsilon) \cap \{u | \operatorname{div} A^{-1/2}u = 0\}$. Indeed, by the definition of $\operatorname{dom}(\mathcal{A}^\varepsilon)$, if $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, $\operatorname{div} g = 0$, and $u, v \in \operatorname{dom}(\mathcal{A}^\varepsilon)$ are such that $\mathcal{A}^\varepsilon u + u = -A^{1/2}g$ and $\mathcal{A}^\varepsilon v + v = -u$, we obtain

$$\int_{\mathbb{R}^3} |u|^2 d\mu^\varepsilon = \int_{\mathbb{R}^3} A^{1/2}g \cdot \overline{v} d\mu^\varepsilon.$$

If $A^{1/2}g$ is orthogonal to $\operatorname{dom}(\mathcal{A}^\varepsilon)$, then $u = 0$, and therefore $A^{1/2}g = 0$, which proves the claim. Furthermore $\mathcal{A}^\varepsilon$ is clearly symmetric and is actually self-adjoint since its defect numbers are zero.

As in previous chapter, all integrals and differential operators, unless indicated otherwise, are understood appropriately with respect to the measure μ. Throughout the chapter we use the notation e_κ for the exponent $\exp(i\kappa \cdot y)$, $y \in Q$, $\kappa \in [-\pi, \pi)^3$, and a similar notation e_θ for the exponent $\exp(i\theta \cdot x)$, $x \in \mathbb{R}^3$, $\theta \in \varepsilon^{-1}[-\pi, \pi)^3$. We denote by $C^\infty_\#$ the set of Q-periodic vector functions in $C^\infty(\mathbb{R}^3)$ and by $C^1_\#$ the set of Q-periodic vector functions in $C^1(\mathbb{R}^3)$. The $\operatorname{curl}\phi$, $\operatorname{curl}(e_\kappa\phi)$ $\operatorname{curl}(e_{\varepsilon\theta}\phi)$ are the classical curls of smooth functions ϕ, $e_\kappa\phi$, $e_{\varepsilon\theta}\phi$.

5.1 Floquet transform

Before we define a notion of the Floquet transform suitable for the analysis of (5.4) and the related family of operator problem, we define the Sobolev spaces of quasiperiodic functions with respect the measure μ.

Definition 5.1.1. *For each $\kappa \in [-\pi, \pi)^3 := Q'$, the space $H^1_{\operatorname{curl} A^{1/2},\kappa}(Q, d\mu)$ is defined as the closure of the set $\{e_\kappa\phi, \operatorname{curl}(e_\kappa A^{1/2}\phi) : \phi \in C^1_\#\}$ in the norm $L^2(Q, d\mu) \oplus L^2(Q, d\mu)$. We use the notation $H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu) = H^1_{\operatorname{curl} A^{1/2}}$ for the space $H^1_{\operatorname{curl} A^{1/2},\kappa}$ with $\kappa = 0$.*

Note that there may be more than one elements in $H^1_{\operatorname{curl} A^{1/2},\kappa}$ with the same first component. Furthermore there is a one-to-one map linking $H^1_{\operatorname{curl} A^{1/2},\kappa}$ and $H^1_{\operatorname{curl} A^{1/2}}$. In fact for any couple $(u, v) \in H^1_{\operatorname{curl} A^{1/2},\kappa}$ the pair $(\overline{e_\kappa}u, \overline{e_\kappa}(v - i\kappa \times A^{1/2}u)) \in H^1_{\operatorname{curl} A^{1/2}}$, which is a consequence of

$$\operatorname{curl}(A^{1/2}\phi_n) = \overline{e_\kappa} \operatorname{curl}(e_\kappa A^{1/2}\phi_n) - i\kappa \times A^{1/2}\phi_n$$

for every $\phi_n \in C^1_\#$ such that $e_\kappa\phi_n \to e_\kappa u$, $\operatorname{curl}(e_\kappa A^{1/2}\phi_n) \to \operatorname{curl}(e_\kappa A^{1/2}u)$. Conversely, for every $(\widetilde{u}, \widetilde{v}) \in H^1_{\operatorname{curl} A^{1/2}}$ one has $\widetilde{v} = \overline{e_\kappa}(v - i\kappa \times A^{1/2}u)$ for some $(u, v) \in H^1_{\operatorname{curl} A^{1/2},\kappa}$.

For every $\kappa \in Q'$ we focus our analysis on the operator $\mathcal{A}_\kappa$ with domain

$$
\begin{aligned}
\mathrm{dom}(\mathcal{A}_\kappa) = \{u \in L^2(Q, d\mu) :\ \exists\, \mathrm{curl}(e_\kappa A^{1/2} u) \text{ such that}\\
\int_Q \mathrm{curl}(e_\kappa A^{1/2} u) \cdot \overline{\mathrm{curl}(e_\kappa A^{1/2} \phi)} d\mu + \int_Q u \cdot \overline{\phi} d\mu = -\int_Q A^{1/2} G \cdot \overline{\phi} d\mu\\
\forall \phi \in H^1_{\mathrm{curl}\, A^{1/2},\kappa},\ \text{for some } G \in L^2(Q, d\mu),\ \overline{e_\kappa}\, \mathrm{div}(e_\kappa G) = 0\},
\end{aligned}
$$

defined by the formula $\mathcal{A}_\kappa u = -A^{1/2} G - u$, where $G \in L^2(Q, d\mu)$ and $u \in \mathrm{dom}(\mathcal{A}_\kappa)$ are linked by the above formula. By an argument similar to the one developed for $\mathcal{A}^\varepsilon$ we infer that its domain is dense in $L^2(Q, d\mu) \cap \{u |\ \overline{e_\kappa}\, \mathrm{div}(e_\kappa A^{-1/2} u) = 0\}$. Furthermore $\mathcal{A}_\kappa$ is clearly symmetric, and is self-adjoint.

In order to transform the problem (5.4), we first recall the definition of the Floquet transformation for functions in $L^2(\mathbb{R}^3, d\mu^\varepsilon)$ (*cf.* with Section 4.1.1). For $\varepsilon > 0$, the ε-Floquet transform $\mathcal{F}_\varepsilon$ is defined for $u \in C_0^\infty(\mathbb{R}^3)$ as:

$$
(\mathcal{F}_\varepsilon u)(\theta, z) = \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \sum_{n \in \mathbb{Z}^3} u(z + \varepsilon n) \exp(-i\varepsilon n \cdot \theta) \quad z \in \varepsilon Q,\ \theta \in \varepsilon^{-1} Q'.
$$

Note that the mapping $\mathcal{F}_\varepsilon$ preserves the norm and can be extended to an isometry from $L^2(\mathbb{R}^3, d\mu^\varepsilon)$ to $L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon)$. The inverse is defined as

$$
(\mathcal{F}_\varepsilon g)^{-1}(z) = \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \int_{\varepsilon^{-1} Q'} g(\theta, z) d\theta \quad g \in L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon).
$$

Observe that $\mathcal{F}_\varepsilon$ is a unitary transform. To obtain the representation for the operator $\mathcal{A}^\varepsilon$, we combine the ε-Floquet transform with the following unitary scaling transform:

$$
\mathcal{T}_\varepsilon h(\theta, y) = \varepsilon^{3/2} h(\theta, \varepsilon y) \quad \theta \in \varepsilon^{-1} Q',\ y \in Q,\ h \in L^2(\varepsilon^{-1} Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon),
$$

$$
(\mathcal{T}_\varepsilon h)^{-1}(\theta, z) = \varepsilon^{-3/2} h(\theta, z/\varepsilon) \quad \theta \in \varepsilon^{-1} Q',\ z \in \varepsilon Q,\ h \in L^2(\varepsilon^{-1} Q' \times Q, d\theta \times d\mu).
$$

Proposition 5.1.2. *For each $\varepsilon > 0$ the following unitary equivalence between the operator $\mathcal{A}^\varepsilon$ and the direct integral of the operator family $\mathcal{A}_\kappa$, $\kappa := \varepsilon\theta$, $\theta \in \varepsilon^{-1} Q'$, holds:*

$$
(\mathcal{A}^\varepsilon + I)^{-1} = \mathcal{F}_\varepsilon^{-1} \mathcal{T}_\varepsilon^{-1} \left(\int_{\varepsilon^{-1} Q'}^{\oplus} e_\kappa (\varepsilon^{-2} \mathcal{A}_\kappa + I)^{-1} \overline{e_\kappa} d\theta \right) \mathcal{T}_\varepsilon \mathcal{F}_\varepsilon.
$$

Sketch of proof. The argument is similar to the one discussed in Chapter 4 for the system of Maxwell equations with zero external currents. Let us consider the solution $(D_\varepsilon, \mathrm{curl}\, A^{1/2} D_\varepsilon) \in H^1_{\mathrm{curl}\, A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$ of problem (5.4) with $g \in C_0^\infty(\mathbb{R}^3)$. For such D_ε

we denote the "periodic amplitude" of its Floquet transform as follows

$$D_\theta^\varepsilon(y) := \overline{e_{\varepsilon\theta}}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon D_\varepsilon = \left(\frac{\varepsilon^2}{2\pi}\right)^{3/2}\sum_{n\in\mathbb{Z}^3} D_\varepsilon(\varepsilon y+\varepsilon n)\exp(-i(\varepsilon y+\varepsilon n)\cdot\theta), \quad y\in Q.$$

For any choice of $\operatorname{curl}(A^{1/2}D_\varepsilon)$, in particular for the one in (5.4), we have that

$$\operatorname{curl}(e_{\varepsilon\theta}A^{1/2}D_\theta^\varepsilon)(y) = \varepsilon\left(\frac{\varepsilon^2}{2\pi}\right)^{3/2}\sum_{n\in\mathbb{Z}^3}\operatorname{curl}(A^{1/2}D_\varepsilon(\varepsilon y+\varepsilon n))\exp(-i\varepsilon n\cdot\theta), \quad y\in Q$$

is a curl of $e_{\varepsilon\theta}D_\theta^\varepsilon$ in sense that $(e_{\varepsilon\theta}D_\theta^\varepsilon, \operatorname{curl}(e_{\varepsilon\theta}A^{1/2}D_\theta^\varepsilon))$ is an element of $H^1_{\operatorname{curl}A^{1/2},\varepsilon\theta}$. Therefore

$$\varepsilon^{-2}\int_Q \operatorname{curl}(e_\kappa A^{1/2}D_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)}d\mu + \int_Q D_\theta^\varepsilon\cdot\overline{\phi}d\mu = -\int_Q A^{1/2}G\cdot\overline{\phi}d\mu \tag{5.7}$$

for all $\forall(e_\kappa\phi, \operatorname{curl}(e_\kappa A^{1/2}\phi))\in H^1_{\operatorname{curl}A^{1/2},\kappa}(Q,d\mu)$. The function G us defined as $G := \overline{e_{\varepsilon\theta}}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon g$, and is such that $\overline{e_\kappa}\operatorname{div}(e_\kappa G)=0$ in the sense

$$\int_Q e_\kappa G\cdot\overline{\nabla(e_\kappa\phi)} = 0 \quad \forall\phi\in C^\infty_\#. \tag{5.8}$$

The density of $C_0^\infty(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3, d\mu^\varepsilon)$ implies the claim. $\square$

In what follows we study the behaviour of the solution D_θ^ε to the problem:

$$\varepsilon^{-2}A^{1/2}\overline{e_\kappa}\operatorname{curl}\operatorname{curl}(e_\kappa A^{1/2}D_\theta^\varepsilon) + D_\theta^\varepsilon = -A^{1/2}G, \quad \varepsilon>0, \quad \kappa\in Q'. \tag{5.9}$$

The function G in $L^2(Q,d\mu)$ is such that $\overline{e_\kappa}\operatorname{div}(e_\kappa G)=0$ in the sense (5.8). The problem (5.9) is understood with the integral identity (5.7).

5.2 Helmholtz decomposition

The Helmholtz, or Hodge, decomposition for square-integrable functions (see e.g. [17, Chapter 2], [29, Chapter 9], [49, Section 3.7]), is an important tool in the analysis of the system of Maxwell equations. In this section we provide a version of such decomposition which takes into account the quasiperiodicity of functions involved, the arbitrary of the measure μ and the geometry of problem (5.9). Note that the main difference with the version of Helmholtz decomposition developed in Chapter 4, is that here the matrix A plays an important role in the subspaces' geometry. Before formulating the next proposition, we refer the reader to the notion of a gradient of a quasiperiodic L^2 function with respect to a measure μ and the associated Sobolev spaces $H^1_\kappa(Q,d\mu)$ introduced in Section 2.1.

In what follows $C^\infty_{\#,0}$ is the set of infinitely smooth Q-periodic functions with zero mean over Q, and $H^1_{\#,0}$ the set of Q-periodic functions in $H^1_{\rm loc}(\mathbb{R}^3, d\mu)$ with zero mean over Q.

We say that a vector $v \in L^2(Q, d\mu)$ is solenoidal, or $\overline{e_\kappa}\,{\rm div}(e_\kappa A^{-1/2})$-free, if

$$\int_Q A^{-1/2} e_\kappa v \cdot \overline{\nabla(e_\kappa \phi)} d\mu = 0 \qquad \forall \phi \in C^\infty_\#. \tag{5.10}$$

Furthermore, we say that a vector $v \in L^2(Q, d\mu)$ is irrotational, or $\overline{e_\kappa}\,{\rm curl}(e_\kappa A^{1/2})$-free, if

$$\int_Q A^{1/2} e_\kappa v \cdot \overline{{\rm curl}\,(e_\kappa \phi)} d\mu = 0 \qquad \forall \phi \in C^\infty_\#. \tag{5.11}$$

Clearly, the (linear) subspaces of $L^2(Q, d\mu)$ solenoidal and irrotational functions are orthogonal.

In order to construct the Helmholtz decomposition we introduce a problem for the scalar function Φ_w with the following proposition.

Proposition 5.2.1. *For any $w \in L^2(Q, d\mu)$ there exists a unique solution $\Phi_w \in H^1_{\#,0}$ to the problem*

$$\overline{e_\kappa}\,{\rm div}\, A^{-1}\nabla(e_\kappa \Phi_w) = \overline{e_\kappa}\,{\rm div}(e_\kappa A^{-1/2} w), \tag{5.12}$$

understood in the sense of the integral identity

$$\int_Q A^{-1}\nabla(e_\kappa \Phi_w) \cdot \overline{\nabla(e_\kappa \phi)} = \int_Q A^{-1/2} e_\kappa w \cdot \overline{\nabla(e_\kappa \phi)} \qquad \forall \phi \in C^\infty_{\#,0}. \tag{5.13}$$

Proof. The left-hand side of equation (5.13) defines a sesquilinear form that is bounded and coercive on $H^1_{\#,0}$. The coercivity follows from the Poincaré inequality (2.11) proved in Section 2.2 for the scalar case. Bearing in mind that the right hand side of (5.13) is a bounded linear functional on $H^1_{\#,0}$, we use the Riesz representation theorem to prove the existence and uniqueness of solution Φ_w. □

For any $v \in L^2(Q, d\mu)$ there exist $c \in \mathbb{C}^3$ and $\psi_c \in H^1_{\#,0}$ such that the element

$$v = A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa \psi_c) + c\big) \tag{5.14}$$

is solenoidal, hence holds

$$\overline{e_\kappa}\,{\rm div}\, A^{-1}\big(\nabla(e_\kappa \psi_c) + e_\kappa c\big) = 0, \tag{5.15}$$

in the sense that (*cf.* (5.10))

$$\int_Q A^{-1}\nabla(e_\kappa\psi_c)\cdot\overline{\nabla(e_\kappa\phi)} = -\int_Q A^{-1}e_\kappa c\cdot\overline{\nabla(e_\kappa\phi)} \quad \forall\phi\in C^\infty_{\#,0}. \tag{5.16}$$

The existence and uniqueness of a solution $\psi_c \in H^1_{\#,0}$ for the problem (5.16) is a direct consequence of Proposition 5.2.1, setting $w = -A^{-1/2}c$ in (5.12)-(5.13).

Writing $\psi_c = \Psi_\kappa \cdot c$ for a vector function Ψ_κ we have

$$A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa\psi_c)+c\big) = A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I\big)c, \tag{5.17}$$

where $(\nabla\Psi_\kappa)_{ij} := (\Psi_\kappa)_{j,i}$, $i,j=1,2,3$, and the vector function Ψ_κ, with components in $H^1_{\#,0}$, is the solution of (*cf.* (5.15))

$$\overline{e_\kappa}\,\mathrm{div}\,A^{-1}\big(\nabla(e_\kappa\Psi_\kappa)+e_\kappa I\big)=0. \tag{5.18}$$

Consider a function $w\in L^2(Q,d\mu)$. Proposition 5.2.1 provides a unique function Φ_w with zero mean such that $w - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_w)$ is solenoidal. We write it as the sum of two elements, $\widetilde{w}$ and $A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa\psi_c)+c\big)$, which yields

$$w = \widetilde{w} + A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa\psi_c)+c\big) + A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_w), \tag{5.19}$$

where $\widetilde{w}$ is solenoidal and

$$\int_Q A^{-1/2}\widetilde{w} = 0, \tag{5.20}$$

$A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa\psi_c)+c\big)$ is solenoidal since ψ_c is the solution of (5.15), and $A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_w)$ is irrotational. It follows that all terms of (5.19) are L^2-orthogonal to each other.

Lemma 5.2.2. *For any function $w\in L^2(Q,d\mu)$, the constant c in the representation (5.19) is given by*

$$c = \bigg(\int_Q A^{-1}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)\bigg)^{-1}\bigg(\int_Q A^{-1/2}w - \int_Q A^{-1}\overline{e_\kappa}\nabla(e_\kappa\Phi_w)\bigg). \tag{5.21}$$

Proof. Starting from (5.19), we take into account the equality (5.17), we multiply by $A^{-1/2}$ and integrate over Q. By the property (5.20), the claim follows. □

5.2.1 Poincaré-type inequality

In what follows we make the following assumption about the measure μ in order to provide a version of the Poincaré inequality for functions in the Sobolev space $H^1_{\mathrm{curl}\,A^{1/2},\kappa}(Q,d\mu)$.

Assumption 5.2.3. *For each $w\in L^2(Q,d\mu)$, define the constant $c=c(w)$ by the* *formula (5.21). There exists $C > 0$ such that for all $\kappa \in Q'$ and $(e_\kappa w, \operatorname{curl}(e_\kappa A^{1/2} w)) \in H^1_{\operatorname{curl} A^{1/2},\kappa}(Q, d\mu)$ one has*

$$\left\| w - A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa \psi_c) + c\big) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_w)\right\|_{L^2(Q,d\mu)} \le C\left\|\operatorname{curl}(e_\kappa A^{1/2} w)\right\|_{L^2(Q,d\mu)} \tag{5.22}$$

Lemma 5.2.4. *For any $\eta \in H^1_{\#,0}$, the zero vector is one of the curls of $A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\eta)$, i.e. one has*

$$\big(A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \eta), 0\big) \in H^1_{\operatorname{curl} A^{1/2},\kappa}(Q, d\mu).$$

Proof. The statement follows from (5.11), indeed $A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\eta)$ is irrotational. In fact we have that

$$\int_Q \nabla(e_\kappa \eta)\cdot \overline{\operatorname{curl}(e_\kappa \phi)} = 0 \quad \forall \phi \in C^\infty_\#,$$

since the curl vanishes on gradient fields. □

5.3 Asymptotic approximation of D^ε_θ

In this section, we present the construction of the asymptotic approximation for the solution of problem (5.9) in order to introduce the main result of the chapter. We state the operator-norm resolvent estimate for the function D^ε_θ and consequently the estimate for the electric induction D_ε in the whole space setting.

5.3.1 The main result of the chapter

Theorem 5.3.1. *Assume a measure μ such that the Poincaré-type inequality* (5.22) *holds, and a real-valued matrix-valued function A such that is continuously differentiable Q-periodic symmetric and positive definite. Then the following estimate holds for the solution D^ε_θ of* (5.9) *with a constant $C > 0$ independent of ε, θ and G:*

$$\left\| D^\varepsilon_\theta - A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big) d^\varepsilon_\theta \right\|_{L^2(Q,d\mu)} \le C\varepsilon \|G\|_{L^2(Q,d\mu)}. \tag{5.23}$$

For each ε, θ, the vector $d^\varepsilon_\theta \in \mathbb{C}^3$ is defined by

$$d^\varepsilon_\theta = -\big(\widehat{A}^{\mathrm{hom}}_{\varepsilon\theta}\big)^{-1}\big(\mathfrak{A}^{\mathrm{hom}}_\theta + I\big)^{-1}\int_Q G. \tag{5.24}$$

Here $\mathfrak{A}^{\mathrm{hom}}_\theta$ is the matrix valued quadratic form

$$\mathfrak{A}^{\mathrm{hom}}_\theta = i\theta \times \big(i\theta \times (\widehat{A}^{\mathrm{hom}}_{\varepsilon\theta})^{-1}\big),$$

with
$$\widehat{A}^{\text{hom}}_{\varepsilon\theta} := \int_Q A^{-1}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big), \tag{5.25}$$
where $\Psi_{\varepsilon\theta}$ *is the solution of* (5.18).

An estimate analogous to (5.23) holds for the transformed electric field $E^{\varepsilon}_{\theta} := A^{1/2}D^{\varepsilon}_{\theta}$. In fact, as a direct consequence of Theorem 5.3.1 we have the following result:

Theorem 5.3.2. *Under the assumptions on the measure* μ *and the coefficient* A *stated in Theorem 5.3.1, the following estimate holds for the transformed electric field* E^{ε}_{θ} *with a constant* $C > 0$ *independent of* ε, θ *and* G*:*

$$\|E^{\varepsilon}_{\theta} - \big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d^{\varepsilon}_{\theta}\|_{L^2(Q,d\mu)} \leq C\varepsilon\|G\|_{L^2(Q,d\mu)}. \tag{5.26}$$

Remark 5.3.3. *Define* N *as the matrix in* $H^1_{\text{curl}}(Q, d\mu)$ *solving the cell problem*

$$\operatorname{curl} A(\operatorname{curl} N + I) = 0, \quad \operatorname{div} N = 0.$$

It can be shown that (see [19, Lemma 4.4])

$$\big(\widehat{A}^{\text{hom}}_0\big)^{-1} = A^{\text{hom}} := \int_Q A(\operatorname{curl} N + I).$$

In what follows we make the following assumption about the homogenised matrix $\widehat{A}^{\text{hom}}_{\kappa}$ defined in (5.25).

Assumption 5.3.4. *There exist constants* $C_1,\ C_2 > 0$*, independent of* $\kappa \in Q'$*, such that the following estimates hold for* $\widehat{A}^{\text{hom}}_{\kappa}$*:*

$$C_1 I \leq \widehat{A}^{\text{hom}}_{\kappa} \leq C_2 I. \tag{5.27}$$

Remark 5.3.5. *Note that the upper bound in* (5.27) *can be proved with an argument similar to the one used in the proof of the classical Voigt-Reiss inequality (see [74, Chapter 1]). In fact, the quadratic form for* $\widehat{A}^{\text{hom}}_{\kappa}$ *is given by*

$$\widehat{A}^{\text{hom}}_{\kappa}\lambda\cdot\lambda := \inf_{\psi_\lambda \in H^1_{\#,0}} \int_Q A^{-1}\big(\overline{e_\kappa}\nabla(e_\kappa\psi) + \lambda\big)\cdot\big(\overline{e_\kappa}\nabla(e_\kappa\psi) + \lambda\big), \quad \lambda \in \mathbb{C}^3, \tag{5.28}$$

By setting $\psi = 0$ *in the integral on the right hand side, this immediately implies the upper bound in* (5.27) *with* $C_2 := \|A^{-1}\|_{L^\infty}$.

To obtain the operator-norm resolvent estimates in the whole space setting for the initial problem (5.4), it remains to apply the inverse Floquet transform to the asymptotic estimate (5.23). Hence, we obtain the following corollary of Theorem 5.3.1.

Corollary 5.3.6. *Suppose that $g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ is a divergence free functions and denote $g^\varepsilon_\theta := \overline{e_\kappa}\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon g^\varepsilon$ so that*

$$\int_Q g^\varepsilon_\theta d\mu = \widehat{g}(\theta,\varepsilon), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \widehat{g}(\theta,\varepsilon) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} g^\varepsilon \overline{e_\theta}\, d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.$$

There exists a constant $C > 0$ such that the following estimate holds for the solution D_ε of (5.4)

$$\left\| D_\varepsilon - (2\pi)^{-3/2}\int_{\mathbb{R}^3} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)\big(\widehat{A}^{\text{hom}}_{\varepsilon\theta}\big)^{-1}\big(\mathfrak{A}^{\text{hom}}_\theta + I\big)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta d\theta \right\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \leq C\varepsilon \|g^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}, \tag{5.29}$$

$\forall \varepsilon > 0$*. Here $\mathfrak{A}^{\text{hom}}_\theta$ is the matrix valued quadratic form defined in* (5.24)*, and $\Psi_{\varepsilon\theta}$ by* (5.18) *for all values $\theta \in \mathbb{R}^3$.*

Proof of Corollary 5.3.6. Throughout the proof we shall drop the superscription ε in g^ε for brevity. Consider D^ε_θ solution of (5.9) with $G = g^\varepsilon_\theta$ on has that

$$\begin{aligned}
&D_\varepsilon - (2\pi)^{-3/2}\int_{\mathbb{R}^3} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d^\varepsilon_\theta d\theta = \\
&\big[\mathcal{F}^{-1}_\varepsilon\mathcal{T}^{-1}_\varepsilon e_{\varepsilon\theta}D^\varepsilon_\theta - \mathcal{F}^{-1}_\varepsilon\mathcal{T}^{-1}_\varepsilon e_{\varepsilon\theta}A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d^\varepsilon_\theta\big] \\
&+ \big[\mathcal{F}^{-1}_\varepsilon\mathcal{T}^{-1}_\varepsilon e_{\varepsilon\theta}A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d^\varepsilon_\theta \\
&- (2\pi)^{-3/2}\int_{\mathbb{R}^3} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)\big(\widehat{A}^{\text{hom}}_{\varepsilon\theta}\big)^{-1}\big(\mathfrak{A}^{\text{hom}}_\theta + I\big)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta\big].
\end{aligned}$$

To prove the corollary we need to analyse the L^2 norm of the above equality. In view of the Theorem 5.3.1 and the unitary property of $\mathcal{F}_\varepsilon$ and $\mathcal{T}_\varepsilon$, we can estimate the first bracket in the right hand side as follows

$$\big\|\mathcal{F}^{-1}_\varepsilon\mathcal{T}^{-1}_\varepsilon e_{\varepsilon\theta}[D^\varepsilon_\theta - A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d^\varepsilon_\theta]\big\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \leq C\varepsilon\big\|g\big\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}. \tag{5.30}$$

Noting that

$$\begin{aligned}
&\mathcal{F}^{-1}_\varepsilon\mathcal{T}^{-1}_\varepsilon e_{\varepsilon\theta}A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d^\varepsilon_\theta = \\
&(2\pi)^{-3/2}\int_{\varepsilon^{-1}Q'} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)\big(\widehat{A}^{\text{hom}}_{\varepsilon\theta}\big)^{-1}(\mathfrak{A}^{\text{hom}}_\theta + I)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta d\theta,
\end{aligned}$$

it remains to analyse

$$(2\pi)^{-3/2}\int_{\mathbb{R}^3\setminus\varepsilon^{-1}Q'} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta})+I\big)\big(\widehat{A}^{\rm hom}_{\varepsilon\theta}\big)^{-1}(\mathfrak{A}^{\rm hom}_{\theta}+I)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta d\theta. \quad (5.31)$$

Using the estimates (5.27) we obtain

$$\sup_{\theta\in\mathbb{R}^3\setminus\varepsilon^{-1}Q'}\Big|\big(\widehat{A}^{\rm hom}_{\varepsilon\theta}\big)^{-1}(\mathfrak{A}^{\rm hom}_{\theta}+I)^{-1}\Big|\leq\frac{C_1^{-1}\varepsilon^2}{C_2^{-1}\pi^2+\varepsilon^2}. \quad (5.32)$$

Using the Parseval identity, the (5.32) and the uniform bound for $\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta})$ obtained by (5.18), we can estimate the L^2 norm of (5.31) by

$$\frac{C_1^{-1}\varepsilon^2}{C_2^{-1}\pi^2+\varepsilon^2}\widetilde{C}\big\|\widehat{g}(\theta,\varepsilon)\big\|_{L^2(\mathbb{R}^3)}\leq\frac{C_1^{-1}\varepsilon^2}{C_2^{-1}\pi^2+\varepsilon^2}\widetilde{C}\|g\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}. \quad (5.33)$$

Combining (5.30) and (5.33), the claim follows. □

5.3.2 Formal interpretation of the main result

The estimate in the whole space (5.29) allows us to approximate the solution D_ε of (5.4), with the function

$$(2\pi)^{-3/2}\int_{\mathbb{R}^3} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta})+I\big)\big(\widehat{A}^{\rm hom}_{\varepsilon\theta}\big)^{-1}\big(\mathfrak{A}^{\rm hom}_{\theta}+I\big)^{-1}\widehat{g}(\theta)e_\theta d\theta. \quad (5.34)$$

It is the correct replacement of the homogenised solution operator. Note that (5.34) is a pseudo-differential operator with two-scale symbol depending on θ and $\varepsilon\theta$.

Here we discuss, from a formal point of view, the meaning of (5.34). This pseudo-differential operator can be always written as a formal series in powers of ε. It is important to note that such series is not rigorous. In fact if we try to truncate it at some order of ε then we lose part of its meaning: this is not an asymptotic series and it is not possible to estimate the reminder. The reason for it resides in the structure of the operator, in fact for every ε we have an integral with respect to θ, which has a role in the estimate (5.29).

Let us analyse the first element of this formal series of infinite order. When $\varepsilon=0$ in (5.34), we have the following standard construction:

$$\begin{aligned}A^{-1/2}(\nabla\Psi+I)\big(\widehat{A}^{\rm hom}_0\big)^{-1}(2\pi)^{-3/2}&\int_{\mathbb{R}^3}\big(\mathfrak{A}^{\rm hom}_{\theta}+I\big)^{-1}\widehat{g}(\theta)e_\theta d\theta\\ &=A^{-1/2}(\nabla\Psi+I)\big(\widehat{A}^{\rm hom}_0\big)^{-1}\big(\mathfrak{A}^{\rm hom}+I\big)^{-1}g.\end{aligned}$$

Here $\mathfrak{A}^{\rm hom}=\operatorname{curl}\operatorname{curl}\big(\widehat{A}^{\rm hom}_0\big)^{-1}$, and Ψ is the vector function with components in $H^1_\#$

solution of the following problem (cf. (5.18))

$$\operatorname{div} A^{-1}(\nabla\Psi + I) = 0, \qquad \int_Q \Psi = 0.$$

Recalling the formal homogenised equation (5.3) for the Maxwell system in the whole space, we have that

$$\big(\operatorname{curl}\operatorname{curl} A^{\rm hom} + I\big)^{-1} g^\varepsilon = \mathcal{D}_\varepsilon^{\rm hom},$$

where $A^{\rm hom} = \big(\widehat{A}_0^{\rm hom}\big)^{-1}$. Hence we have that, for $\varepsilon = 0$, (5.34) is

$$A^{-1/2}(\nabla\Psi + I)\big(\widehat{A}_0^{\rm hom}\big)^{-1}\mathcal{D}_\varepsilon^{\rm hom}. \tag{5.35}$$

Note that this expression contains both the solution of the formal homogenised equation, and rapidly oscillating terms. It has the same structure as the limit term obtained for the electric displacement in Chapter 4 (see [20]), for the setting of system of the Maxwell equations with zero external currents.

High-order terms in (5.34) are solutions of some singular perturbed problems, and they all contribute to the leading-order term of the series. In fact high-order terms have a non trivial dependence on θ for every ε, and the presence of θ can not be ignored.

The expression for the approximating solution operator in the estimate for D_ε exhibits a dependence on $y \in Q$ and ε. This does not appear in the case of the system of Maxwell equations with zero external current, where there is no ε-dependent corrector term in the estimates.

Results about norm resolvent estimates for the system of Maxwell equation with external current and unitary magnetic permeability have been obtained by Birman and Suslina in [10] for the setting of Lebesgue measure. They construct a special corrector depending on ε in order to obtain estimates. In our case the structure of (5.34) is a consequence of the Poicaré-type inequality and the Helmholtz decomposition, which provide a representation for the space of square integrable functions. With our approach, it is possible to have an explicit and more compact homogenised solution operator which contains the standard construction (5.35) and an infinite series depending on ε.

5.3.3 The asymptotic approximation

We now proceed to the proof of Theorem 5.3.1. For each $\theta \in \varepsilon^{-1}Q'$, $\varepsilon > 0$, we write the following approximation for the solution of (5.9):

$$D_\theta^\varepsilon := U_\theta^\varepsilon + z_\theta^\varepsilon, \tag{5.36}$$

where

$$U_\theta^\varepsilon = A^{-1/2}\big(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa) + I\big)d_\theta^\varepsilon + \varepsilon^2 R_\theta^\varepsilon. \tag{5.37}$$

The vector $d_\theta^\varepsilon \in \mathbb{C}^3$ is defined in (5.24), and Ψ_κ is the solution of (5.18). The function $R_\theta^\varepsilon \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ is defined as the solution to the problem

$$
\begin{aligned}
&A^{1/2}\overline{e_\kappa} \operatorname{curl}\operatorname{curl}(e_\kappa A^{1/2} R_\theta^\varepsilon) + \varepsilon^2 A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) + \varepsilon^2 A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R_\theta^\varepsilon}) \\
&= -A^{1/2}G - \varepsilon^{-2}A^{1/2}\overline{e_\kappa}\operatorname{curl} e_\kappa\big(\mathrm{i}\kappa \times d_\theta^\varepsilon\big) - A^{-1/2}\big(e_\kappa \nabla(e_\kappa \Psi_\kappa) + I\big)d_\theta^\varepsilon =: \mathcal{H}_\theta^\varepsilon, \qquad (5.38)
\end{aligned}
$$

with $\mathcal{H}_\theta^\varepsilon \in (H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu))^*$, and $\psi_{R_\theta^\varepsilon}$, $c_{R_\theta^\varepsilon}$ and $\Phi_{R_\theta^\varepsilon}$ are defined as in the decomposition (5.19). The equation (5.38) is understood in the sense of the integral identity

$$
\begin{aligned}
&\int_Q \operatorname{curl}(e_\kappa A^{1/2} R_\theta^\varepsilon) \cdot \overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)} + \varepsilon^2 \int_Q A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) \cdot \overline{\phi} \\
&\qquad + \varepsilon^2 \int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R_\theta^\varepsilon}) \cdot \overline{\phi} = \langle \mathcal{H}_\theta^\varepsilon, \phi\rangle \quad \forall \phi \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu). \qquad (5.39)
\end{aligned}
$$

Proposition 5.3.7. *There exists a unique solution $R_\theta^\varepsilon \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ for the equation* (5.38).

Proof. Using the decomposition (5.19) for ϕ in the equation (5.39), and using the orthogonality between the elements of such decomposition, we have:

$$
\int_Q A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) \cdot \overline{\phi} = \int_Q A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) \cdot \overline{A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \psi_\phi) + c_\phi)},
$$

and

$$
\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R_\theta^\varepsilon}) \cdot \overline{\phi} = \int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R_\theta^\varepsilon}) \cdot \overline{A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_\phi)}.
$$

The proof of existence and uniqueness is a consequence of Lax-Millgram theorem applied to the skew-symmetric sesquilinear form

$$
\begin{aligned}
b(u, v) = &\int_Q \operatorname{curl}(e_\kappa A^{1/2} u) \cdot \overline{\operatorname{curl}(e_\kappa A^{1/2} v)} + \varepsilon^2 \int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_u) \cdot \overline{A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_v)} \\
&+ \varepsilon^2 \int_Q A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \psi_u) + c_u) \cdot \overline{A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \psi_v) + c_v)},
\end{aligned}
$$

for $u, v \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$, ψ_u, ψ_v solving (5.15), and Φ_u, Φ_v solutions of (5.12). Note that $b(u, v)$ is bounded and coercive on $H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$. The coercivity follows from the Poincaré-type inequality (5.22). $\square$

The right hand side of (5.39) can be rewritten as follows:

$$
\begin{aligned}
&\langle \mathcal{H}_\theta^\varepsilon, \phi\rangle \\
&= -\int_Q \Big(A^{1/2}G + A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d_\theta^\varepsilon\Big)\cdot\overline{\phi} - \varepsilon^{-1}\int_Q e_\kappa(i\theta\times d_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)} \\
&= -\int_Q \Big(A^{1/2}G + A^{1/2}i\theta\times(i\theta\times d_\theta^\varepsilon) + A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d_\theta^\varepsilon\Big)\cdot\overline{\phi} \qquad \forall\phi\in H^1_{\operatorname{curl}A^{1/2},\kappa}.
\end{aligned}
$$

In the last equality we use that for $\phi\in C^1_\#$ on has

$$\overline{e_\kappa}\operatorname{curl}(e_\kappa A^{1/2}\phi) = \operatorname{curl}(A^{1/2}\phi) + i\kappa\times A^{1/2}\phi$$

and, furthermore, $\int_Q i\theta\times d_\theta^\varepsilon\cdot\overline{\operatorname{curl}(A^{1/2}\phi)} = 0$. It follows that

$$\mathcal{H}_\theta^\varepsilon = -A^{1/2}G - A^{1/2}i\theta\times(i\theta\times d_\theta^\varepsilon) - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d_\theta^\varepsilon. \tag{5.40}$$

5.3.4 Properties of $\mathcal{H}_\theta^\varepsilon$

In order to prove Theorem 5.3.1, we will achieve estimates for R_θ^ε. The tool we use to obtain such estimate, is the Poincaré-type inequality (5.22). In order to do it, we prove the following identity:

$$\langle\mathcal{H}_\theta^\varepsilon, R_\theta^\varepsilon\rangle = \langle\mathcal{H}_\theta^\varepsilon, \widetilde{R}_\theta^\varepsilon\rangle,$$

where $\widetilde{R}_\theta^\varepsilon$ is defined as in the decomposition (5.19).
To provide such identity, we are interested in two properties for $\mathcal{H}_\theta^\varepsilon$. First of all

$$\langle\mathcal{H}_\theta^\varepsilon, A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\phi)\rangle = 0 \quad \forall\phi\in H^1_{\#,0}. \tag{5.41}$$

Starting with the definition of $\mathcal{H}_\theta^\varepsilon$ in (5.38), we have

$$\langle\mathcal{H}_\theta^\varepsilon, A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\phi)\rangle = \int_Q e_\kappa G\cdot\overline{\nabla(e_\kappa\phi)} + \int_Q A^{-1}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d_\theta^\varepsilon\cdot\overline{\overline{e_\kappa}\nabla(e_\kappa\phi)},$$

since $A^{1/2}\overline{e_\kappa}\operatorname{curl}e_\kappa(i\theta\times d_\theta^\varepsilon)$ is solenoidal. The first integral is zero using that $\overline{e_\kappa}\operatorname{div}(e_\kappa G) = 0$ (see (5.8)). The second integral is null by equation (5.18) with $c = d_\theta^\varepsilon$.
The second property we want to prove for $\mathcal{H}_\theta^\varepsilon$ is

$$\langle\mathcal{H}_\theta^\varepsilon, A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_c)+c)\rangle = 0 \quad \forall\psi_c\in H^1_{\#,0},\ c\in\mathbb{C}^3. \tag{5.42}$$

By linearity, we obtain

$$\langle\mathcal{H}_\theta^\varepsilon, A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_c)+c)\rangle = \langle\mathcal{H}_\theta^\varepsilon, A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\psi_c)\rangle + \langle\mathcal{H}_\theta^\varepsilon, A^{-1/2}c\rangle.$$

Using (5.41), it remains to examine the expression

$$\langle \mathcal{H}_\theta^\varepsilon, A^{-1/2}c\rangle = -\int_Q G\cdot c - \int_Q A^{-1}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d_\theta^\varepsilon\cdot c - \int_Q i\theta\times(i\theta\times d_\theta^\varepsilon)\cdot c.$$

Noticing that the expression on the right hand side enters the weak form of the equation (5.24), solved by d_θ^ε, we conclude that (5.42) is satisfied.

5.4 Estimate for $\varepsilon^2 R_\theta^\varepsilon$

Theorem 5.4.1. *There exists $C>0$ such that for all $\varepsilon>0$ and $\theta\in\varepsilon^{-1}Q'$, the solution of the equation* (5.38) *satisfies:*

$$\|R_\theta^\varepsilon - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon})+c_{R_\theta^\varepsilon}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\|_{L^2(Q,d\mu)} \le C\|G\|_{L^2(Q,d\mu)}, \quad (5.43)$$

$$\|A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon})+c_{R_\theta^\varepsilon}) + A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\|_{L^2(Q,d\mu)} \le C\varepsilon^{-1}\|G\|_{L^2(Q,d\mu)}. \quad (5.44)$$

Proof. Suppose that $\phi_n\in C^1_\#$ converging to R_θ^ε in $L^2(Q,d\mu)$ are such that $\operatorname{curl}(e_\kappa A^{1/2}\phi_n)$ converge to $\operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon)$ in $L^2(Q,d\mu)$. Let us write (5.39) with $\phi=\phi_n$. By (5.40) one has

$$\begin{aligned}
&\int_Q \operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi_n)} + \varepsilon^2\int_Q A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon})+c_{R_\theta^\varepsilon})\cdot\overline{\phi_n}\\
&\quad+\varepsilon^2\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\cdot\overline{\phi_n}\\
&= -\int_Q\Big(A^{1/2}G + A^{1/2}i\theta\times i\theta\times d_\theta^\varepsilon + A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d_\theta^\varepsilon\Big)\cdot\overline{\phi_n}. \qquad (5.45)
\end{aligned}$$

Passing to the limit $n\to\infty$ we can write the left hand side of (5.45) as a quadratic form. To this end, recalling the decomposition (5.19) for R_θ^ε and the related orthogonality conditions, we have that

$$\int_Q A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon})+c_{R_\theta^\varepsilon})\cdot\overline{R_\theta^\varepsilon} = \int_Q \big|A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon})+c_{R_\theta^\varepsilon})\big|^2,$$

and

$$\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\cdot\overline{R_\theta^\varepsilon} = \int_Q\big|A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\big|^2.$$

Hence

$$\int_Q \operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon)} + \varepsilon^2\int_Q\big|A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon})+c_{R_\theta^\varepsilon})\big|^2$$

$$+\varepsilon^2\int_Q \left|A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta})\right|^2 = \langle\mathcal{H}^\varepsilon_\theta, R^\varepsilon_\theta\rangle. \tag{5.46}$$

Furthermore we use the properties (5.41) and (5.42) of $\mathcal{H}^\varepsilon_\theta$ to rewrite the left-hand side of (5.46) as

$$\begin{aligned}\int_Q &\operatorname{curl}(e_\kappa A^{1/2}R^\varepsilon_\theta)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}R^\varepsilon_\theta)} + \varepsilon^2\int_Q \left|A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R^\varepsilon_\theta}) + c_{R^\varepsilon_\theta})\right|^2\\ &+\varepsilon^2\int_Q\left|A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta})\right|^2\\ &= -\int_Q\Big(A^{1/2}G + A^{1/2}i\theta\times i\theta\times d^\varepsilon_\theta + A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d^\varepsilon_\theta\Big)\cdot\overline{\widetilde{R^\varepsilon_\theta}}.\end{aligned} \tag{5.47}$$

Using the Poincaré-type inequality (5.22) for R^ε_θ and the definition of d^ε_θ (5.24) the following estimate holds

$$\|\operatorname{curl}(e_\kappa A^{1/2}R^\varepsilon_\theta)\|_{L^2(Q,d\mu)} \le C\|G\|_{L^2(Q,d\mu)}. \tag{5.48}$$

Combining estimate (5.48) and the Poincaré-type inequality (5.22) one obtains (5.43). The same estimates and equation (5.47) imply (5.44). □

Corollary 5.4.2. *There exists a constant $C>0$ such that the following estimate holds uniformly in ε, θ and G:*

$$\|U^\varepsilon_\theta - A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa\Psi_\kappa)+I)d^\varepsilon_\theta\|_{L^2(Q,d\mu)} \le C\varepsilon\|G\|_{L^2(Q,d\mu)}.$$

5.4.1 Conclusion of the convergence estimate

Proposition 5.4.3. *There exists $C>0$ such that the function z^ε_θ defined in* (5.36) *satisfies the estimate*

$$\|z^\varepsilon_\theta\|_{L^2(Q,d\mu)} \le C\varepsilon\|G\|_{L^2(Q,d\mu)}. \tag{5.49}$$

Proof. The function $z^\varepsilon_\theta \in H^1_{\operatorname{curl}A^{1/2}}(Q,d\mu)$ solves the problem

$$\begin{aligned}&\varepsilon^{-2}A^{1/2}\overline{e_\kappa}\operatorname{curl}\operatorname{curl}e_\kappa A^{1/2}z^\varepsilon_\theta + z^\varepsilon_\theta\\ &\quad= -\varepsilon^2\big(R^\varepsilon_\theta - A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa\psi_{R^\varepsilon_\theta}) + c_{R^\varepsilon_\theta}) - A^{1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta})\big).\end{aligned} \tag{5.50}$$

Using z^ε_θ as a test function in the integral formulation of equation (5.50), one has

$$\varepsilon^{-2}\int_Q\operatorname{curl}(e_\kappa A^{1/2}z^\varepsilon_\theta)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}z^\varepsilon_\theta)} + \int_Q|z^\varepsilon_\theta|^2 = -\varepsilon^2\int_Q\widetilde{R^\varepsilon_\theta}\cdot\overline{z^\varepsilon_\theta}. \tag{5.51}$$

Applying the Hölder inequality, the Poincaré-type inequality (5.22) and the estimate (5.48) one has (5.49). □

Note that using (5.49) on the second term on the left hand side of (5.51), and a combination of Hölder inequality and (5.48) on the right hand side of (5.51), one has

$$\| \operatorname{curl}(e_\kappa A^{1/2} z_\theta^\varepsilon)\|_{L^2(Q,d\mu)} \leq C\varepsilon^2 \|G\|_{L^2(Q,d\mu)}. \tag{5.52}$$

Corollary 5.4.2, together with Proposition 5.4.3, give us (5.23). In fact

$$\begin{aligned}
&\big\|D_\theta^\varepsilon - A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d_\theta^\varepsilon\big\|_{L^2(Q,d\mu)}\\
&\leq \|U_\theta^\varepsilon - A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)d_\theta^\varepsilon\|_{L^2(Q,d\mu)} + \|z_\theta^\varepsilon\|_{L^2(Q,d\mu)},
\end{aligned}$$

which concludes the proof of Theorem 5.3.1.

5.4.2 Estimate for magnetic field and magnetic induction

It is possible to obtain estimates for the magnetic field and magnetic induction starting from equation (5.9). In fact, the transformed problem can be written in terms of the Maxwell system:

$$\begin{cases}\varepsilon^{-1}\overline{e_\kappa}\operatorname{curl}(e_\kappa A^{1/2}D_\theta^\varepsilon) + B_\theta^\varepsilon = 0,\\ \varepsilon^{-1}A^{1/2}\overline{e_\kappa}\operatorname{curl}(e_\kappa B_\theta^\varepsilon) - D_\theta^\varepsilon = A^{1/2}G.\end{cases} \tag{5.53}$$

Here $\overline{e_\kappa}\operatorname{div}(e_\kappa A^{-1/2}D_\theta^\varepsilon) = 0$, and $B_\theta^\varepsilon := \overline{e_\kappa}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon\mathcal{B}_\varepsilon$ is the transformed magnetic induction such that $\overline{e_\kappa}\operatorname{div}(e_\kappa B_\theta^\varepsilon) = 0$. In this case the transformed magnetic field H_θ^ε coincides with B_θ^ε.

To find the approximation for B_θ^ε we use the approximation of D_θ^ε (5.36) and we plug it in system (5.53). We obtain that

$$\begin{aligned}
B_\theta^\varepsilon &= \varepsilon^{-1}\overline{e_\kappa}\operatorname{curl}\big(e_\kappa A^{1/2}(A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa) + I)d_\theta^\varepsilon + \varepsilon^2 R_\theta^\varepsilon + z_\theta^\varepsilon)\big)\\
&= i\theta\times d_\theta^\varepsilon + \varepsilon\overline{e_\kappa}\operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon) + \varepsilon^{-1}\overline{e_\kappa}\operatorname{curl}(e_\kappa A^{1/2}z_\theta^\varepsilon).
\end{aligned}$$

In the last equality we use Lemma 5.2.4. Here d_θ^ε solves (5.24), R_θ^ε solves (5.38) and z_θ^ε is the solution of (5.50). As a consequence of estimates (5.48) and (5.52) we can state the following result.

Theorem 5.4.4. *Under the assumptions on the measure μ and the coefficient A stated in Theorem 5.3.1, there exists $C > 0$ independent of θ, ε and G such that the following estimate holds for the transformed magnetic induction B_θ^ε (and consequently for the transformed magnetic field H_θ^ε):*

$$\|B_\theta^\varepsilon - i\theta\times d_\theta^\varepsilon\|_{L^2(Q,d\mu)} \leq \varepsilon C\|G\|_{L^2(Q,d\mu)}. \tag{5.54}$$

It remains to apply the inverse Floquet transform to (5.54) to obtain the norm-resolvent estimate in the whole space setting for the first component $\mathcal{B}_\varepsilon$ of the solution of (5.2).

Corollary 5.4.5. *Let $g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ and denote $g^\varepsilon_\theta := \overline{e_\kappa}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon g^\varepsilon$ so that*

$$\int_Q g^\varepsilon_\theta d\mu = \widehat{g}(\theta,\varepsilon), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \widehat{g}(\theta,\varepsilon) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} g^\varepsilon \overline{e_\theta}\, d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.$$

There exists a constant $C > 0$ such that the following estimate holds for $\mathcal{B}_\varepsilon$ solution of (5.2)

$$\left\|\mathcal{B}_\varepsilon - \operatorname{curl}\left((2\pi)^{-3/2}\int_{\mathbb{R}^3}\big(\widehat{A}^{\rm hom}_{\varepsilon\theta}\big)^{-1}\big(\mathfrak{A}^{\rm hom}_\theta + I\big)^{-1}\widehat{g}(\theta)e_\theta d\theta\right)\right\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \le C\varepsilon\|g^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}, \tag{5.55}$$

$\forall\varepsilon > 0$. Here $\mathfrak{A}^{hom}_\theta$ is the matrix valued quadratic form given by (5.24).

As discussed in Section 5.3.2, here the estimate (5.55) allows us to approximate B_ε with the pseudo-differential operator

$$\operatorname{curl}\left((2\pi)^{-3/2}\int_{\mathbb{R}^3}\big(\widehat{A}^{\rm hom}_{\varepsilon\theta}\big)^{-1}\big(\mathfrak{A}^{\rm hom}_\theta + I\big)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta d\theta\right). \tag{5.56}$$

It is the correct replacement of what we can think as the homogenised solution operator. As done for (5.34), it is known that (5.56) can be written as a formal series in powers of ε. Note that when $\varepsilon = 0$ we have the following standard construction

$$\operatorname{curl}\big(\widehat{A}^{\rm hom}_0\big)^{-1}(2\pi)^{-3/2}\int_{\mathbb{R}^3}\big(\mathfrak{A}^{\rm hom}_\theta + I\big)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta d\theta = \operatorname{curl}\big(\widehat{A}^{\rm hom}_0\big)^{-1}\big(\mathfrak{A}^{\rm hom} + I\big)^{-1}g^\varepsilon,$$

where $\mathfrak{A}^{\rm hom} = \operatorname{curl}\operatorname{curl}\big(\widehat{A}^{\rm hom}_0\big)^{-1}$. Recalling the formal homogenised problem for the Maxwell system in the whole space (5.3), we have that for $\varepsilon = 0$ the pseudo-differential operator (5.56) is

$$\operatorname{curl} A^{\rm hom}\mathcal{D}^{\rm hom}_\varepsilon = \mathcal{B}^{\rm hom}_\varepsilon,$$

where $A^{\rm hom} = \big(\widehat{A}^{\rm hom}_0\big)^{-1}$. Note that, when $\varepsilon = 0$, we obtain an approximation term that is similar to the one we have in the estimates for the magnetic induction in Chapter 4, for the Maxwell system with relative magnetic permeability set to unity and zero external currents. In the present chapter, however, the high-order terms of (5.56) are all contributive for the limit term in (5.55). The behaviour of the solution operator in the estimate for $\mathcal{B}_\varepsilon$ is linked to the solution of the homogenised Maxwell system (5.3) $\mathcal{B}^{\rm hom}_\varepsilon$, and to the high-order terms of (5.56) depending on ε.

Chapter 6

Operator-norm homogenisation estimates for Maxwell equations on periodic singular structures: the general system

Introduction

In this chapter we conclude the analysis of the system of Maxwell equations on singular periodic structures. This result can be found in the second part of the preprint [21] by Cherednichenko and D'Onofrio. The focus of the chapter is the general system of Maxwell equations, where the magnetic permeability is an arbitrary matrix-valued function. This analysis is closely linked to Chapter 5, where we proved operator-norm resolvent estimates the system of Maxwell equations in the case with non-zero external currents and magnetic permeability set to unity, which has the form

$$A^{1/2}\operatorname{curl}\operatorname{curl}A^{1/2}u^\varepsilon+u^\varepsilon=-A^{1/2}g^\varepsilon,\quad g^\varepsilon\in L^2(\mathbb{R}^3,d\mu^\varepsilon),\quad \operatorname{div}g^\varepsilon=0,\quad \varepsilon>0.\quad (6.1)$$

The problem (6.1) is an intermediate step between the case of Maxwell system with zero external currents analysed in Chapter 4 and the general Maxwell system object of this chapter. The approach to study the problem (6.1), is based on the analysis of the family of operators, parametrised by the quasimomentum θ, obtained applying the Floquet transform to the original problem. We produced an approximation for the solution in powers of ε, which contains rapidly oscillating elements in the leading order term, and we obtained an estimate uniform in θ for the reminder. The tools developed in Chapter 5 for the non-zero current case are essential in what follows. In fact, the Helmholtz decomposition (5.19) allows us to analyse the general Maxwell system, because it takes into account the quasiperiodicity of the functions involved in the Floquet transform, and the fact that they are solenoidal and irrotational in sense of definitions (5.10) and (5.11).

The full system of Maxwell equations has been analysed by Suslina in [58], [60] and [67] in the setting of Lebesgue measure. Here we tackle this problem with the approach developed in Chapter 5, again in the setting of periodic singular structures.

We consider, (see Introduction in Chapters 4 and 5) a Q-periodic Borel measure μ in $\mathbb{R}^3$ where $Q := [0,1)^3$, such that $\mu(Q) = 1$. For each $\varepsilon > 0$ we define μ^ε as $\mu^\varepsilon(B) = \varepsilon^3 \mu(\varepsilon^{-1}B)$ for every $B \subset \mathbb{R}^3$ Borel set.

The general system of Maxwell equations written in terms of displacement vectors has the form (*cf.* (5.2)):

$$\begin{cases} \operatorname{curl}(A(\cdot/\varepsilon)\mathcal{D}_\varepsilon) + \mathcal{B}_\varepsilon = f^\varepsilon, \\ \operatorname{curl}(\widetilde{A}(\cdot/\varepsilon)\mathcal{B}_\varepsilon) - \mathcal{D}_\varepsilon = g^\varepsilon, \end{cases} \tag{6.2}$$

where g^ε and f^ε are divergence-free vectorial functions in $L^2(\mathbb{R}^3, d\mu^\varepsilon)$. In (6.2) $\mathcal{B}_\varepsilon$ is the magnetic induction, and $\mathcal{D}_\varepsilon$ is the electric displacement. The inverse of the relative dielectric permittivity A is a real-valued, continuously differentiable, Q-periodic matrix function, which is assumed to be symmetric and positive definite. The inverse of the relative magnetic permeability $\widetilde{A}$, is a real-valued, bounded, Q-periodic matrix function, assumed to be symmetric and positive definite.

Our goal here is to prove operator-norm resolvent estimates for the solutions of (6.2) for ε small, with the solution to a suitable homogenised problem which serves as a replacement to the formally suggested system

$$\begin{cases} \operatorname{curl}(A^{\mathrm{hom}}\mathcal{D}_\varepsilon^{\mathrm{hom}}) + \mathcal{B}_\varepsilon^{\mathrm{hom}} = f^\varepsilon, \\ \operatorname{curl}(\widetilde{A}^{\mathrm{hom}}\mathcal{B}_\varepsilon^{\mathrm{hom}}) - \mathcal{D}_\varepsilon^{\mathrm{hom}} = g^\varepsilon, \end{cases} \tag{6.3}$$

where A^{hom} and $\widetilde{A}^{\mathrm{hom}}$ are the matrices of "effective" homogenised coefficients. As discussed at the end of Chapter 3, the solution of the standard formal limit system (6.3) turns out to be not operator-norm close to the solution of the original problem as $\varepsilon \to 0$. We will show, in the same spirit of Chapter 5, that the correct homogenised system is in some sense a singular perturbation of (6.3), which involves a ε-dependent pseudodifferential operator.

In what follows we study, without loss of generality, the system (6.2) with $f^\varepsilon = 0$.

We start our analysis rewriting the system (6.2) as a symmetric form, following the approach of [58, Section 3]. We define $A^{1/2}\mathcal{D}_\varepsilon := D_\varepsilon$, and we obtain an equivalent formulation for (6.2):

$$A^{1/2} \operatorname{curl} \widetilde{A}\big(\operatorname{curl}(A^{1/2}D_\varepsilon)\big) + D_\varepsilon = -A^{1/2}g^\varepsilon, \quad g^\varepsilon \in L^2(Q, d\mu^\varepsilon), \quad \operatorname{div} g^\varepsilon = 0. \tag{6.4}$$

The solution of (6.4) is understood as the pair $(D_\varepsilon, \operatorname{curl}(A^{1/2}D_\varepsilon)) \in H^1_{\operatorname{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$

such that

$$\int_{\mathbb{R}^3} \widetilde{A}\,\mathrm{curl}(A^{1/2}D_\varepsilon)\cdot\overline{\mathrm{curl}(A^{1/2}\phi)}+\int_{\mathbb{R}^3} D_\varepsilon\cdot\overline{\phi} = -\int_{\mathbb{R}^3} A^{1/2}g^\varepsilon\cdot\overline{\phi} \quad \forall\phi\in H^1_{\mathrm{curl}\,A^{1/2}}(\mathbb{R}^3,d\mu^\varepsilon). \tag{6.5}$$

For the definition of $H^1_{\mathrm{curl}\,A^{1/2}}$ we refer to Chapter 5. Note that for every $\varepsilon>0$ the left-hand side of (6.5) is an equivalent inner product on $H^1_{\mathrm{curl}\,A^{1/2}}(\mathbb{R}^3,d\mu^\varepsilon)$ and the right-hand side is linear bounded functional on $H^1_{\mathrm{curl}\,A^{1/2}}(\mathbb{R}^3,d\mu^\varepsilon)$. Hence, existence and uniqueness of solution to (6.4) are a consequence of the Riesz representation theorem.

In what follows we study the resolvent of the operator $\widetilde{\mathcal{A}}^\varepsilon$ with domain

$$\begin{aligned}\mathrm{dom}(\widetilde{\mathcal{A}}^\varepsilon) = \{u\in L^2(\mathbb{R}^3,d\mu^\varepsilon):\ &\exists\,\mathrm{curl}\,A^{1/2}u \text{ such that}\\ &\int_{\mathbb{R}^3}\widetilde{A}\,\mathrm{curl}(A^{1/2}u)\cdot\overline{\mathrm{curl}(A^{1/2}\phi)}+\int_{\mathbb{R}^3}u\cdot\overline{\phi} = -\int_{\mathbb{R}^3}A^{1/2}g\cdot\overline{\phi}\quad\forall\phi\in H^1_{\mathrm{curl}\,A^{1/2}}(\mathbb{R}^3,d\mu^\varepsilon),\\ &\text{for some } g\in L^2(\mathbb{R}^3,d\mu^\varepsilon),\ \mathrm{div}\,g=0\},\end{aligned}$$

defined by $\widetilde{\mathcal{A}}^\varepsilon D_\varepsilon = -A^{1/2}g - D_\varepsilon$, where $g\in L^2(\mathbb{R}^3,d\mu^\varepsilon)$, $\mathrm{div}\,g=0$ and $D_\varepsilon\in\mathrm{dom}(\widetilde{\mathcal{A}}^\varepsilon)$ are linked as in the above formula. Note that $\widetilde{\mathcal{A}}^\varepsilon$ is clearly symmetric. Furthermore, the $\mathrm{dom}(\widetilde{\mathcal{A}}^\varepsilon)$ is dense in $L^2(\mathbb{R}^3,d\mu^\varepsilon)\cap\{u|\,\mathrm{div}\,A^{-1/2}u=0\}$, and $\widetilde{\mathcal{A}}^\varepsilon$ is self-adjoint (*cf.* Chapter 5).

6.1 Floquet transform

In this section we present the family of operator problems obtained from (6.4) via the Floquet transform defined in Section 5.1, and we provide the problem unitary equivalent to (6.4), which is the object of our analysis.

We introduce, for each $\kappa\in Q'$, the operator $\widetilde{\mathcal{A}}_\kappa$ with domain

$$\begin{aligned}\mathrm{dom}(\widetilde{\mathcal{A}}_\kappa) = \{u\in L^2(Q,d\mu):\ &\exists\,\mathrm{curl}(e_\kappa A^{1/2}u) \text{ such that}\\ &\int_Q\widetilde{A}\,\mathrm{curl}(e_\kappa A^{1/2}u)\cdot\overline{\mathrm{curl}(e_\kappa A^{1/2}\phi)}d\mu+\int_Q u\cdot\overline{\phi}d\mu = -\int_Q A^{1/2}G\cdot\overline{\phi}\quad\forall\phi\in H^1_{\mathrm{curl}\,A^{1/2},\kappa}\\ &\text{for some } G\in L^2(Q,d\mu),\ \overline{e_\kappa}\,\mathrm{div}(e_\kappa G)=0\},\end{aligned}$$

is defined by the formula $\widetilde{\mathcal{A}}_\kappa u = -A^{1/2}G-u$. Note that $\widetilde{\mathcal{A}}_\kappa$ is clearly symmetric. Furthermore the $\mathrm{dom}(\widetilde{\mathcal{A}}_\kappa)$ is dense in $L^2(Q,d\mu)\cap\{u|\ \mathrm{div}\,A^{-1/2}u=0\}$, and $\widetilde{\mathcal{A}}_\kappa$ is self-adjoint.

In the spirit of Section 5.1, we state the following proposition, bearing in mind the definitions of the ε-Floquet transform $\mathcal{F}_\varepsilon$ and the scaling transform $\mathcal{T}_\varepsilon$.

Proposition 6.1.1. *For each $\varepsilon>0$ the following unitary equivalence between the operator $\widetilde{\mathcal{A}}^\varepsilon$ and the direct integral of the operator family $\widetilde{\mathcal{A}}_\kappa$, for $\kappa:=\varepsilon\theta$, $\theta\in\varepsilon^{-1}Q'$*

holds:

$$(\widetilde{\mathcal{A}}^{\varepsilon}+I)^{-1}=\mathcal{F}_{\varepsilon}^{-1}\mathcal{T}_{\varepsilon}^{-1}\int_{\varepsilon^{-1}Q'}^{\oplus}e_{\kappa}(\varepsilon^{-2}\widetilde{\mathcal{A}}_{\kappa}+I)^{-1}\overline{e_{\kappa}}d\theta\mathcal{T}_{\varepsilon}\mathcal{F}_{\varepsilon}.$$

Therefore, in the present chapter we study the behaviour of $D_{\theta}^{\varepsilon}:=\overline{e_{\kappa}}\mathcal{T}_{\varepsilon}\mathcal{F}_{\varepsilon}D_{\varepsilon}$, solution to the problem

$$\varepsilon^{-2}A^{1/2}\overline{e_{\kappa}}\operatorname{curl}\widetilde{A}\big(\operatorname{curl}(e_{\kappa}A^{1/2}D_{\theta}^{\varepsilon})\big)+D_{\theta}^{\varepsilon}=-A^{1/2}G,\quad \varepsilon>0,\quad \kappa\in Q'. \tag{6.6}$$

where $G\in L^2(Q,d\mu)$ is a function such that $\overline{e_{\kappa}}\operatorname{div}(e_{\kappa}G)=0$ in sense of (5.8). The solution to the problem (6.6) is understood as the pair

$$(e_{\kappa}D_{\theta}^{\varepsilon},\operatorname{curl}(e_{\kappa}A^{1/2}D_{\theta}^{\varepsilon}))\in H^1_{\operatorname{curl}A^{1/2},\kappa}(Q,d\mu)$$

such that

$$\varepsilon^{-2}\int_Q\widetilde{A}\operatorname{curl}(e_{\kappa}A^{1/2}D_{\theta}^{\varepsilon})\cdot\overline{\operatorname{curl}(e_{\kappa}A^{1/2}\phi)}d\mu+\int_Q D_{\theta}^{\varepsilon}\cdot\overline{\phi}d\mu=-\int_Q A^{1/2}G\cdot\overline{\phi}d\mu$$
$$\forall(e_{\kappa}\phi,\operatorname{curl}(e_{\kappa}A^{1/2}\phi)\in H^1_{\operatorname{curl}A^{1/2},\kappa}(Q,d\mu).$$

6.2 Asymptotic approximation of D_{θ}^{ε}

In this section we present the operator-norm resolvent estimates for the solution D_{θ}^{ε} of equation (6.6), and its asymptotic approximation in powers of ε.

We assume throughout this chapter, that the measure μ is such that the embedding

$$H^1_{\operatorname{curl}A^{1/2}}(Q,d\mu)\cap\{w:\ \operatorname{div}(A^{-1/2}w)=0\}\subset L^2(Q,d\mu) \tag{6.7}$$

is compact.

6.2.1 The main result of the chapter

In order to write an asymptotic approximation for D_{θ}^{ε}, we consider the following "cell problem" for the matrix $\widetilde{N}\in H^1_{\operatorname{curl}A^{1/2}}(Q,d\mu)$:

$$\begin{cases}A^{1/2}\operatorname{curl}\widetilde{A}\operatorname{curl}(A^{1/2}\widetilde{N})=-A^{1/2}\operatorname{curl}\widetilde{A},\\ \operatorname{div}(A^{-1/2}\widetilde{N})=0,\quad \int_Q A^{1/2}\widetilde{N}=0.\end{cases} \tag{6.8}$$

Proposition 6.2.1. *There exists a unique solution $\widetilde{N}\in H^1_{\operatorname{curl}A^{1/2}}(Q,d\mu)$ for the equation* (6.8)*, understood in the sense of the integral identity*

$$\int_Q\widetilde{A}\operatorname{curl}(A^{1/2}\widetilde{N})\cdot\overline{\operatorname{curl}(A^{1/2}\phi)}d\mu=-\int_Q\widetilde{A}\overline{\operatorname{curl}(A^{1/2}\phi)} \tag{6.9}$$

$\forall \phi \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ *such that* $\int_Q A^{1/2}\phi = 0$.

Proof. It follows from the compactness of the embedding (6.7) that the sesquilinear form

$$\int_Q \widetilde{A} \operatorname{curl}(A^{1/2}u) \cdot \overline{\operatorname{curl}(A^{1/2}v)} d\mu,$$
$$u, v \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu) \cap \left\{ u : \operatorname{div} A^{1/2}u = 0, \int_Q A^{1/2}u = 0 \right\},$$

is coercive. Note that it is also continuous. The existence and uniqueness of solution of (6.9) are a consequence of the Riesz representation theorem. □

We assume the measure μ such that, for any $\operatorname{curl} A^{1/2}$-free matrix V, there exist a vector function Ψ with components in $H^1_{\#,0}$, and a constant matrix $a \in \mathbb{C}^{3\times 3}$ such that

$$V = A^{-1/2}(\nabla\Psi + a), \tag{6.10}$$

where $(\nabla\Psi)_{ij} = \Psi_{j,i}$.

We denote with M the subspace of $L^2(Q, d\mu)$ consisting of matrices which are both $\operatorname{div} A^{-1/2}$-free and $\operatorname{curl} A^{1/2}$-free (*cf.* with definitions (5.10) and (5.11) with $\kappa = 0$). For any $V \in M$, there exist $a \in \mathbb{C}^{3\times 3}$ and a vector function $\Psi \in H^1_{\#,0}(Q, d\mu)$ such that

$$\operatorname{div} A^{-1}(\nabla\Psi + a) = 0,$$

in the sense that

$$\int_Q A^{-1}\nabla\Psi \cdot \overline{\nabla\varphi} = -\int_Q A^{-1}a \cdot \overline{\nabla\varphi} \qquad \forall \varphi \in C^\infty_{\#,0}. \tag{6.11}$$

Proposition 6.2.2. *For any* $a \in \mathbb{C}^{3\times 3}$ *there exists a unique* $\Psi \in H^1_{\#,0}$ *solving* (6.11).

Proof. We assuming the measure μ such that the embedding $H^1_\#(Q, d\mu) \subset L^2(Q, d\mu)$ is compact (see Section 2.2), it follows that the sesquilinear form

$$\int_Q A^{-1}\nabla v \cdot \overline{\nabla u}, \qquad (v, \nabla v),\ (u, \nabla u) \in H^1_{\#,0}$$

is bounded and coercive, and defines an equivalent inner product on $H^1_{\#,0}$. Noting that (6.11) is a linear bounded functional on $H^1_{\#,0}$, the claim follows by the Riesz representation theorem. □

We are now ready to state the main result for the general system of Maxwell equations.

Theorem 6.2.3. *Assume a measure μ such that the Poincaré-type inequality* (5.22) *holds, the embedding* (6.7) *is compact, and such that any* $\operatorname{curl} A^{1/2}$*-free function $v \in H^1_{\operatorname{curl} A^{1/2}}$ has the form $v = A^{-1/2}(\nabla\psi + c)$, for a function $\psi \in H^1_{\#,0}$ and a constant $c \in \mathbb{C}^3$. Furthermore, assume a real-valued matrix-function A that is continuously differentiable Q-periodic symmetric and positive definite, and a real-valued matrix-function $\widetilde{A}$ that is bounded Q-periodic symmetric and positive definite.*

Then, there exists a constant $C > 0$ independent of ε, θ and G, such that the following estimate holds for D^ε_θ solution of (6.6)*:*

$$\left\| D^\varepsilon_\theta - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa) + I)d^\varepsilon_\theta \right\|_{L^2(Q,d\mu)} \le C\varepsilon\|G\|_{L^2(Q,d\mu)}. \tag{6.12}$$

The vector $d^\varepsilon_\theta \in \mathbb{C}^3$ is defined as

$$d^\varepsilon_\theta = -\big(\widehat{A}^{\mathrm{hom}}_{\varepsilon\theta}\big)^{-1}\big(\widetilde{\mathfrak{A}}^{\mathrm{hom}}_\theta + I\big)^{-1}\int_Q G, \tag{6.13}$$

where $\widetilde{\mathfrak{A}}^{\mathrm{hom}}_\theta$ is the matrix-valued quadratic form given by the equation

$$\big(i\theta \times \widetilde{A}^{\mathrm{hom}}(i\theta \times (\widehat{A}^{\mathrm{hom}}_{\varepsilon\theta})^{-1}) + I\big)\ \widehat{A}^{\mathrm{hom}}_{\varepsilon\theta} d^\varepsilon_\theta = -\int_Q G,$$

with

$$\widetilde{A}^{\mathrm{hom}} := \int_Q \widetilde{A}(\operatorname{curl}(A^{1/2}\widetilde{N}) + I), \qquad \widehat{A}^{\mathrm{hom}}_{\varepsilon\theta} := \int_Q A^{-1}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa) + I),$$

for $\widetilde{N}$ solution of (6.8) *and Ψ_κ solution of* (5.18).

An estimate analogous to (6.12) holds for the transformed electric field $E^\varepsilon_\theta := A^{1/2}D^\varepsilon_\theta$, as a direct consequence of Theorem 6.2.3.

Theorem 6.2.4. *Under the assumptions on the measure μ and the coefficients A, $\widetilde{A}$ stated in Theorem 6.2.3, there exists a constant $C > 0$ independent of ε, θ and G such that for the transformed electric field E^ε_θ holds the following estimate:*

$$\left\| E^\varepsilon_\theta - (\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa) + I)d^\varepsilon_\theta \right\|_{L^2(Q,d\mu)} \le C\varepsilon\|G\|_{L^2(Q,d\mu)}, \tag{6.14}$$

with d^ε_θ defined in (6.13)*, and Ψ_κ solution of* (5.18).

Applying back the Floquet transform to the estimate (6.12) (analogously to (6.14)) one obtains the following operator-norm resolvent estimate in the whole space setting.

Corollary 6.2.5. *Let $g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ and denote $g^\varepsilon_\theta := \overline{e_\kappa}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon g^\varepsilon$ so that*

$$\int_Q g^\varepsilon_\theta d\mu = \widehat{g}(\theta,\varepsilon), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \widehat{g}(\theta,\varepsilon) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} g^\varepsilon \overline{e_\theta}\, d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.$$

There exists a constant $C > 0$ such that the following estimate holds for D_ε solution of (6.4)

$$\left\| D_\varepsilon - (2\pi)^{-3/2}\int_{\mathbb{R}^3} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)\big(\widehat{A}^{\rm hom}_{\varepsilon\theta}\big)^{-1}\big(\widetilde{\mathfrak{A}}^{\rm hom}_\theta + I\big)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta d\theta \right\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \le C\varepsilon\|g^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}, \tag{6.15}$$

$\forall \varepsilon > 0$. *Here $\widetilde{\mathfrak{A}}^{\rm hom}_\theta$ is the matrix valued quadratic form defined with* (6.13)*, and $\Psi_{\varepsilon\theta}$ by* (5.18) *for all values $\theta \in \mathbb{R}^3$.*

The estimate (6.15) has the same structure of (5.29), the one obtained in Chapter 5 for the Maxwell system with magnetic permeability set to unity. The difference between (6.15) and (5.29) is in the definition of $\widetilde{\mathfrak{A}}^{\rm hom}_\theta$, in fact, in (6.15) appears the matrix $\widetilde{A}^{\rm hom}$. However, it is a constant matrix independent on $\varepsilon\theta$, and does not influence the meaning of the pseudo-differential operator

$$(2\pi)^{-3/2}\int_{\mathbb{R}^3} A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta}) + I\big)\big(\widehat{A}^{\rm hom}_{\varepsilon\theta}\big)^{-1}\big(\widetilde{\mathfrak{A}}^{\rm hom}_\theta + I\big)^{-1}\widehat{g}(\theta,\varepsilon)e_\theta d\theta. \tag{6.16}$$

Note that (6.16) is the correct replacement of the homogenised solution operator for the full Maxwell system. For the formal interpretation of (6.16), see Section 5.3.2 where the meaning of the pseudo-differential operator is discussed for the system with unitary magnetic permeability.

6.2.2 The asymptotic approximation

We proceed now to the proof of Theorem 6.2.3. For each $\theta \in \varepsilon^{-1}Q'$, $\varepsilon > 0$, we consider the following approximation for the vector function D^ε_θ solution of (6.6):

$$D^\varepsilon_\theta := U^\varepsilon_\theta + z^\varepsilon_\theta, \tag{6.17}$$

where

$$U^\varepsilon_\theta = A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa) + I)d^\varepsilon_\theta + \varepsilon N(i\theta \times d^\varepsilon_\theta) + \varepsilon^2 R^\varepsilon_\theta. \tag{6.18}$$

Here the matrix $N \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ is defined as

$$N := \widetilde{N} + A^{-1/2}(\nabla\Psi + a_\theta), \tag{6.19}$$

where $\widetilde{N}$ is the solution of (6.8), and $A^{-1/2}(\nabla\Psi + a_\theta)$ is en element in M. Note that the vector function $\Psi \in H^1_{\#,0}$ is the unique solution of equation (6.11). Furthermore, the constant matrix $a_\theta \in \mathbb{C}^{3\times 3}$ is chosen such that

$$\int_Q i\theta \times \widetilde{A}(i\theta \times a_\theta(i\theta \times d^\varepsilon_\theta)) = -\int_Q i\theta \times \widetilde{A}(i\theta \times (A^{1/2}\widetilde{N} + \nabla\Psi)(i\theta \times d^\varepsilon_\theta)), \qquad (6.20)$$

that is

$$\int_Q i\theta \times \widetilde{A}(i\theta \times A^{1/2}N(i\theta \times d^\varepsilon_\theta)) = 0.$$

The constant matrix a_θ is such that for every $\eta \in \Theta^\perp := \{\eta \in \mathbb{C}^3 \mid \eta \cdot \theta = 0\}$, one has $P_{\Theta^\perp}(a_\theta \eta) \neq 0$, where $P_{\Theta^\perp}$ is the orthogonal projection on $\Theta^\perp$.

In the following proposition we prove that there is at least a unique matrix in $\mathbb{C}^{3\times 3}$ satisfying the property (6.20).

Proposition 6.2.6. *There exists a unique $\widetilde{a}_\theta \in \mathbb{C}^{3\times 3}$ such that*

$$\widetilde{a}_\theta \eta \cdot \theta = 0, \qquad \widetilde{a}_\theta \theta = 0, \qquad (6.21)$$

and

$$\int_Q i\theta \times \widetilde{A}(i\theta \times \widetilde{a}_\theta \eta) = -\int_Q i\theta \times \widetilde{A}(i\theta \times (A^{1/2}\widetilde{N} + \nabla\Psi)\eta) \quad \forall \eta \in \Theta^\perp. \qquad (6.22)$$

Proof. The identity (6.22) is equivalent to a linear system for the representation of the matrix $\widetilde{a}_\theta$ in the basis $\{\theta/|\theta|, e_1^\perp, e_2^\perp\}$, for any orthogonal basis $\{e_1^\perp, e_2^\perp\}$ of $\Theta^\perp$. This system is uniquely solvable subject to conditions (6.21) for any right-hand side, if and only if the solution to the related homogeneous system is zero. This is verified noticing that, if

$$\int_Q i\theta \times \widetilde{A}(i\theta \times \widetilde{a}_\theta \eta) = 0 \quad \forall \eta \in \Theta^\perp,$$

then, in particular,

$$\int_Q \widetilde{A}(i\theta \times \widetilde{a}_\theta \eta) \cdot (i\theta \times \widetilde{a}_\theta \eta) = 0 \quad \forall \eta \in \Theta^\perp.$$

We have that $\widetilde{A}$ is positive definite, hence, from the last identity we deduce that $i\theta \times \widetilde{a}_\theta \eta = 0$ and therefore $\widetilde{a}_\theta \eta = 0$ by the first condition in (6.21). Now, by the second condition in (6.21) we obtain that $\widetilde{a}_\theta = 0$ as required. $\square$

Remark 6.2.7. *The set $\Theta^\perp$ can be characterised as $\Theta^\perp = \{\theta \times c,\ c \in \mathbb{C}^3\}$ (cf. Lemma 4.4.5).*

The term $R_\theta^\varepsilon \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ in (6.18), is the solution of the following problem

$$\begin{aligned}
&A^{1/2}\overline{e_\kappa}\operatorname{curl}\widetilde{A}\operatorname{curl} e_\kappa A^{1/2}R_\theta^\varepsilon + \varepsilon^2 A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) + \varepsilon^2 A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon}) = \\
&-A^{1/2}G - \varepsilon^{-2}A^{1/2}\overline{e_\kappa}\operatorname{curl}\widetilde{A}e_\kappa(i\kappa\times d_\theta^\varepsilon) - \varepsilon^{-1}A^{1/2}\overline{e_\kappa}\operatorname{curl}\widetilde{A}\operatorname{curl} e_\kappa(A^{1/2}N(i\theta\times d_\theta^\varepsilon)) \\
&-A^{-1/2}\overline{e_\kappa}(\nabla(e_\kappa\Psi_\kappa) + I)d_\theta^\varepsilon =: \mathcal{H}_\theta^\varepsilon \in (H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu))^*, \qquad (6.23)
\end{aligned}$$

where $\psi_{R_\theta^\varepsilon}$, $c_{R_\theta^\varepsilon}$ and $\Phi_{R_\theta^\varepsilon}$ are defined as in the decomposition (5.19). Such problem is understood in the sense of the integral identity:

$$\begin{aligned}
&\int_Q \widetilde{A}\operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)} + \varepsilon^2\int_Q A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon})\cdot\overline{\phi} \\
&\quad + \varepsilon^2\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\cdot\overline{\phi} = \langle\mathcal{H}_\theta^\varepsilon, \phi\rangle \qquad \forall\phi\in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu). \qquad (6.24)
\end{aligned}$$

Existence and uniqueness of solution $R_\theta^\varepsilon \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ for the equation (6.24) can be proved as in Proposition 5.3.7. The right-hand side of (6.24) is understood as

$$\begin{aligned}
&\langle\mathcal{H}_\theta^\varepsilon, \phi\rangle \\
&= -\int_Q\left(A^{1/2}G + A^{-1/2}\overline{e_\kappa}(\nabla(e_\kappa\Psi_\kappa) + I)d_\theta^\varepsilon\right)\cdot\overline{\phi} - \varepsilon^{-1}\int_Q\widetilde{A}e_\kappa(i\theta\times d_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)} \\
&-\varepsilon^{-1}\int_Q\widetilde{A}\operatorname{curl} e_\kappa(A^{1/2}N(i\theta\times d_\theta^\varepsilon))\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)} \qquad \forall\phi\in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu).
\end{aligned}$$

Setting

$$\operatorname{curl} e_\kappa(A^{1/2}N(i\theta\times d_\theta^\varepsilon)) = e_\kappa(\operatorname{curl} A^{1/2}N(i\theta\times d_\theta^\varepsilon) + i\kappa\times A^{1/2}N(i\theta\times d_\theta^\varepsilon)),$$

using the equation (6.9) and the fact that $\int_Q(i\theta\times d_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(A^{1/2}\phi)} = 0$, we rewrite $\mathcal{H}_\theta^\varepsilon$ as

$$\begin{aligned}
\langle\mathcal{H}_\theta^\varepsilon, \phi\rangle = &-\int_Q\Big(A^{1/2}\big(G + i\theta\times\widetilde{A}(i\theta\times d_\theta^\varepsilon) + i\theta\times\widetilde{A}\operatorname{curl}(A^{1/2}N(i\theta\times d_\theta^\varepsilon)) \\
&+\varepsilon i\theta\times\widetilde{A}(i\theta\times A^{1/2}N(i\theta\times d_\theta^\varepsilon)\big) + A^{-1/2}\overline{e_\kappa}(\nabla(e_\kappa\Psi_\kappa) + I)d_\theta^\varepsilon\Big)\cdot\overline{\phi} \\
&-\int_Q\widetilde{A}i\theta\times(A^{1/2}N(i\theta\times d_\theta^\varepsilon))\cdot\overline{\operatorname{curl} A^{1/2}\phi} \qquad \forall\phi\in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu).
\end{aligned}$$

Hence we can formally write $\mathcal{H}_\theta^\varepsilon$ as:

$$\begin{aligned}\mathcal{H}_\theta^\varepsilon =& - A^{1/2}G - A^{1/2} i\theta \times \widetilde{A}(i\theta \times d_\theta^\varepsilon) - A^{-1/2}\overline{e_\kappa}(\nabla(e_\kappa \Psi_\kappa) + I)d_\theta^\varepsilon \\ &- A^{1/2}\operatorname{curl}\widetilde{A}(i\theta \times A^{1/2}N(i\theta \times d_\theta^\varepsilon)) - A^{1/2} i\theta \times \widetilde{A}\operatorname{curl}(A^{1/2}N(i\theta \times d_\theta^\varepsilon)) \\ &- \varepsilon A^{1/2} i\theta \times \widetilde{A} i\theta \times (A^{1/2}N(i\theta \times d_\theta^\varepsilon)). \end{aligned} \tag{6.25}$$

6.2.3 Properties of $\mathcal{H}_\theta^\varepsilon$

In order to prove the estimates for R_θ^ε, our main tool is the Poincaré-type inequality (5.22). To take advantage of such inequality in the estimates for the right-hand side of (6.23), we prove the identity

$$\langle \mathcal{H}_\theta^\varepsilon, R_\theta^\varepsilon \rangle = \langle \mathcal{H}_\theta^\varepsilon, \widetilde{R}_\theta^\varepsilon \rangle,$$

where $\widetilde{R}_\theta^\varepsilon$ is defined as in the decomposition (5.19). In order to do it, we check two properties for $\mathcal{H}_\theta^\varepsilon$. The first one is:

$$\langle \mathcal{H}_\theta^\varepsilon, A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \phi)\rangle = 0 \quad \forall \phi \in H^1_{\#,0}. \tag{6.26}$$

In fact, starting from definition (6.23) for $\mathcal{H}_\theta^\varepsilon$, we have that (6.26) holds since $\overline{e_\kappa}\operatorname{div}(e_\kappa G) = 0$ and using the equation (5.18) with $c = d_\theta^\varepsilon$ for for the vector function Ψ_κ.

The second property for $\mathcal{H}_\theta^\varepsilon$ is:

$$\langle \mathcal{H}_\theta^\varepsilon, A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa \psi_c) + c)\rangle = 0 \quad \forall \psi_c \in H^1_{\#,0},\ c \in \mathbb{C}^3. \tag{6.27}$$

By linearity we have that

$$\langle \mathcal{H}_\theta^\varepsilon, A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa \psi_c) + c)\rangle = \langle \mathcal{H}_\theta^\varepsilon, A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \psi_c)\rangle + \langle \mathcal{H}_\theta^\varepsilon, A^{-1/2}c\rangle.$$

Hence using the property (6.26), and the formulation (6.25) of $\mathcal{H}_\theta^\varepsilon$, it remains to analyse

$$\begin{aligned}\langle \mathcal{H}_\theta^\varepsilon, A^{-1/2}c\rangle =& -\int_Q G\cdot c - \int_Q A^{-1}(\overline{e_\kappa}\nabla(e_\kappa \Psi_\kappa) + I)d_\theta^\varepsilon \cdot c - \int_Q i\theta \times \widetilde{A}(i\theta \times d_\theta^\varepsilon)\cdot c \\ &- \int_Q i\theta \times \widetilde{A}\operatorname{curl}(A^{1/2}N(i\theta \times d_\theta^\varepsilon))\cdot c - \varepsilon \int_Q i\theta \times \widetilde{A} i\theta \times (A^{1/2}N(i\theta \times d_\theta^\varepsilon))\cdot c.\end{aligned}$$

By the property (6.20), we have that

$$\varepsilon \int_Q i\theta \times \widetilde{A} i\theta \times (A^{1/2}N(i\theta \times d_\theta^\varepsilon))\cdot c = 0.$$

Furthermore,

$$i\theta \times \int_Q \widetilde{A}(\operatorname{curl} A^{1/2}N + I)(i\theta \times d_\theta^\varepsilon)\cdot c + \int_Q A^{-1}(\overline{e_\kappa}\nabla(e_\kappa \Psi_\kappa) + I)d_\theta^\varepsilon \cdot c + \int_Q G\cdot c = 0,$$

since it is the weak formulation of equation (6.13) solved by d_θ^ε. Hence we have that (6.27) holds.

6.3 Estimate for $\varepsilon^2 R_\theta^\varepsilon$

Theorem 6.3.1. *There exists a constant $C>0$ such that for all $\varepsilon>0$ and $\theta\in\varepsilon^{-1}Q'$, the solution of the equation (5.38) satisfies the following estimates:*

$$\left\|R_\theta^\varepsilon - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\right\|_{L^2(Q,d\mu)} \le C\|G\|_{L^2(Q,d\mu)}, \quad (6.28)$$

$$\left\|A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) + A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\right\|_{L^2(Q,d\mu)} \le C\varepsilon^{-1}\|G\|_{L^2(Q,d\mu)}. \quad (6.29)$$

Proof. Suppose $\phi_n\in C^1_\#$ converging to R_θ^ε in $L^2(Q,d\mu)$ are such that $\operatorname{curl}(A^{1/2}\phi_n)$ converge to $\operatorname{curl}(A^{1/2}R_\theta^\varepsilon)$ in $L^2(Q,d\mu)$. Setting ϕ_n as test function in equation (6.24) one has

$$\begin{aligned}
&\int_Q \widetilde{A}\operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi_n)} + \varepsilon^2\int_Q A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon})\cdot\overline{\phi_n}\\
&+\varepsilon^2\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\cdot\overline{\phi_n}\\
&=\langle\mathcal{H}_\theta^\varepsilon,\phi_n\rangle = \langle\mathcal{H}_\theta^\varepsilon,\phi_n - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\rangle,
\end{aligned}$$

where in the last equality we use the properties (6.26) and (6.27) for $\mathcal{H}_\theta^\varepsilon$. Using the identity

$$\begin{aligned}
&\operatorname{curl}A^{1/2}(\phi_n - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})) \qquad (6.30)\\
&\quad= i\kappa\times A^{1/2}(\phi_n - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon}))\\
&\quad+\overline{e_\kappa}\operatorname{curl}e_\kappa A^{1/2}(\phi_n - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})),
\end{aligned}$$

we can rewrite the above equation as

$$\begin{aligned}
&\int_Q \widetilde{A}\operatorname{curl}(e_\kappa A^{1/2}R_\theta^\varepsilon)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi_n)} + \varepsilon^2\int_Q A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon})\cdot\overline{\phi_n}\\
&+\varepsilon^2\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\cdot\overline{\phi_n} = -\int_Q\Big(A^{1/2}\big(G + i\theta\times\widetilde{A}(i\theta\times d_\theta^\varepsilon)\\
&+i\theta\times\widetilde{A}\operatorname{curl}(A^{1/2}N(i\theta\times d_\theta^\varepsilon))\big) + A^{-1/2}\overline{e_\kappa}(\nabla(e_\kappa\Psi_\kappa)+I)d_\theta^\varepsilon\Big)\\
&\cdot\overline{\Big(\phi_n - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R_\theta^\varepsilon})\Big)} - \int_Q e_\kappa\widetilde{A}i\theta\times(A^{1/2}N(i\theta\times d_\theta^\varepsilon))
\end{aligned}$$

$$\cdot \operatorname{curl}\left(e_\kappa A^{1/2}(\phi_n - A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R^\varepsilon_\theta}) + c_{R^\varepsilon_\theta}) - A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta}))\right). \tag{6.31}$$

When $n \to \infty$ we use the decomposition (5.19) and the related orthogonality conditions (*cf.* with Theorem 5.4.1) to obtain a quadratic form in the right-hand side of equation (6.31), and we obtain that

$$\begin{aligned}&\int_Q \widetilde{A}\operatorname{curl}(e_\kappa A^{1/2}R^\varepsilon_\theta)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}R^\varepsilon_\theta)} + \varepsilon^2\int_Q \left|A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{R^\varepsilon_\theta}) + c_{R^\varepsilon_\theta})\right|^2\\&+\varepsilon^2\int_Q\left|A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{R^\varepsilon_\theta})\right|^2 = -\int_Q\Big(A^{1/2}\big(G + i\theta\times\widetilde{A}(i\theta\times d^\varepsilon_\theta)\\&+ i\theta\times\widetilde{A}\operatorname{curl}(A^{1/2}N(i\theta\times d^\varepsilon_\theta))\big) + A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa) + I)d^\varepsilon_\theta\Big)\cdot\overline{\widetilde{R^\varepsilon_\theta}}\\&-\int_Q e_\kappa\widetilde{A}i\theta\times(A^{1/2}N(i\theta\times d^\varepsilon_\theta))\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\widetilde{R^\varepsilon_\theta})}.\end{aligned}\tag{6.32}$$

In order to estimate last term in the right hand side of (6.32), we consider $\xi^\varepsilon_\theta \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ solution of

$$\begin{aligned}A^{1/2}\overline{e_\kappa}\operatorname{curl}\widetilde{A}\operatorname{curl} e_\kappa A^{1/2}\xi^\varepsilon_\theta + \varepsilon^2 A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{\xi^\varepsilon_\theta}) + c_{\xi^\varepsilon_\theta}) + \varepsilon^2 A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{\xi^\varepsilon_\theta})\\ = A^{1/2}\overline{e_\kappa}\operatorname{curl}\widetilde{A}e_\kappa(i\theta\times A^{1/2}N(i\theta\times d^\varepsilon_\theta)),\end{aligned}\tag{6.33}$$

understood as the integral identity

$$\begin{aligned}&\int_Q \widetilde{A}\operatorname{curl}(e_\kappa A^{1/2}\xi^\varepsilon_\theta)\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)} + \varepsilon^2\int_Q A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa\psi_{\xi^\varepsilon_\theta}) + c_{\xi^\varepsilon_\theta})\cdot\overline{\phi}\\&+\varepsilon^2\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa\Phi_{\xi^\varepsilon_\theta})\cdot\overline{\phi} = \int_Q e_\kappa\widetilde{A}i\theta\times(A^{1/2}N(i\theta\times d^\varepsilon_\theta))\cdot\overline{\operatorname{curl}(e_\kappa A^{1/2}\phi)}\end{aligned}\tag{6.34}$$

$\forall\phi\in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$. Here $\psi_{\xi^\varepsilon_\theta}$, $c_{\xi^\varepsilon_\theta}$ and $\Phi_{\xi^\varepsilon_\theta}$ are defined as in the decomposition (5.19). Existence and uniqueness of solution $\xi^\varepsilon_\theta \in H^1_{\operatorname{curl} A^{1/2}}(Q, d\mu)$ follow from the argument used in Proposition 5.3.7. Testing equation (6.33) with ξ^ε_θ we obtain that

$$\|\operatorname{curl}(e_\kappa A^{1/2}\xi^\varepsilon_\theta)\|_{L^2(Q,d\mu)} \le C\|G\|_{L^2(Q,d\mu)}. \tag{6.35}$$

To rewrite the right hand side of equation (6.32), we test equation (6.33) with $\widetilde{R^\varepsilon_\theta}$. We have that

$$\int_Q \widetilde{A} e_\kappa i\theta \times (A^{1/2} N(i\theta \times d_\theta^\varepsilon)) \cdot \overline{\operatorname{curl}(e_\kappa A^{1/2} \widetilde{R_\theta^\varepsilon})} = \int_Q \widetilde{A} \operatorname{curl}(e_\kappa A^{1/2} \xi_\theta^\varepsilon) \cdot \overline{\operatorname{curl}(e_\kappa A^{1/2} \widetilde{R_\theta^\varepsilon})}$$
$$+\varepsilon^2 \int_Q A^{-1/2}(\overline{e_\kappa} \nabla(e_\kappa \psi_{\xi_\theta^\varepsilon}) + c_{\xi_\theta^\varepsilon}) \cdot \overline{\widetilde{R_\theta^\varepsilon}} + \varepsilon^2 \int_Q A^{-1/2} \overline{e_\kappa} \nabla(e_\kappa \Phi_{\xi_\theta^\varepsilon}) \cdot \overline{\widetilde{R_\theta^\varepsilon}}. \tag{6.36}$$

Taking into account the orthogonality of the elements of decomposition (5.19), we have that

$$\int_Q A^{-1/2}(\overline{e_\kappa} \nabla(e_\kappa \psi_{\xi_\theta^\varepsilon}) + c_{\xi_\theta^\varepsilon}) \cdot \overline{\widetilde{R_\theta^\varepsilon}} = 0, \quad \int_Q A^{-1/2} \overline{e_\kappa} \nabla(e_\kappa \Phi_{\xi_\theta^\varepsilon}) \cdot \overline{\widetilde{R_\theta^\varepsilon}} = 0.$$

In order to rewrite the remaining expression in the right hand side of (6.36), we use ξ_θ^ε as test function in equation (6.24). Note that, for an arbitrary measure μ there may be different elements in $H^1_{\operatorname{curl} A^{1/2}}$ with the same first component. Though, for the solution ξ_θ^ε to (6.33) there exists a natural choice of $\operatorname{curl}(A^{1/2}\xi_\theta^\varepsilon)$. In fact, consider sequences ψ_n, $\phi_n \in C^1_\#$ converging to ξ_θ^ε in $L^2(Q, d\mu)$ such that

$$\operatorname{curl}(e_\kappa A^{1/2}\phi_n) \to \operatorname{curl}(e_\kappa A^{1/2}\xi_\theta^\varepsilon), \quad \operatorname{curl}(e_\kappa A^{1/2}\psi_n) \to \operatorname{curl}(e_\kappa A^{1/2}\xi_\theta^\varepsilon).$$

The difference $\operatorname{curl}(e_\kappa A^{1/2}\phi_n) - \operatorname{curl}(e_\kappa A^{1/2}\psi_n)$ converges to zero, and so does $\operatorname{curl}(A^{1/2}\phi_n) - \operatorname{curl}(A^{1/2}\psi_n)$. Henceforth, we denote with $\operatorname{curl}(A^{1/2}\xi_\theta^\varepsilon)$ the common L^2-limit of $\operatorname{curl}(A^{1/2}\phi_n)$ for a sequence $\phi_n \in C^1_\#$ with the above properties.

The unique choice of $\operatorname{curl}(A^{1/2}\xi_\theta^\varepsilon)$ allows us to write

$$\int_Q \widetilde{A} \operatorname{curl}(e_\kappa A^{1/2}\widetilde{R_\theta^\varepsilon}) \cdot \overline{\operatorname{curl}(e_\kappa A^{1/2}\xi_\theta^\varepsilon)} + \varepsilon^2 \int_Q A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa \psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) \cdot \overline{\xi_\theta^\varepsilon}$$
$$+\varepsilon^2 \int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R_\theta^\varepsilon}) \cdot \overline{\xi_\theta^\varepsilon} = \langle \mathcal{H}_\theta^\varepsilon, \xi_\theta^\varepsilon \rangle = \langle \mathcal{H}_\theta^\varepsilon, \widetilde{\xi_\theta^\varepsilon} \rangle,$$

where in the last equality we use the properties (6.26) and (6.27) of $\mathcal{H}_\theta^\varepsilon$.

By the orthogonality of elements in Helmholtz decomposition (5.19), we have

$$\int_Q A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa \psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) \cdot \overline{\xi_\theta^\varepsilon} = \int_Q A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa \psi_{R_\theta^\varepsilon}) + c_{R_\theta^\varepsilon}) \cdot \overline{A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa \psi_{\xi_\theta^\varepsilon}) + c_{\xi_\theta^\varepsilon})},$$

and

$$\int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R_\theta^\varepsilon}) \cdot \overline{\xi_\theta^\varepsilon} = \int_Q A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R_\theta^\varepsilon}) \cdot \overline{A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{\xi_\theta^\varepsilon})}.$$

Hence, the right hand side of (6.36) can be written as

$$\begin{aligned}\int_Q &\widetilde{A} e_\kappa i\theta \times (A^{1/2} N(i\theta \times d^\varepsilon_\theta)) \cdot \overline{\operatorname{curl}(e_\kappa A^{1/2} \widetilde{R^\varepsilon_\theta})} \\ &= \overline{\langle \mathcal{H}^\varepsilon_\theta, \widetilde{\xi^\varepsilon_\theta} \rangle} - \varepsilon^2 \int_Q A^{-1/2} \overline{e_\kappa} \nabla (e_\kappa \Phi_{\xi^\varepsilon_\theta}) \cdot \overline{A^{-1/2} \overline{e_\kappa} \nabla (e_\kappa \Phi_{R^\varepsilon_\theta})} \\ &\quad - \varepsilon^2 \int_Q A^{-1/2} (\overline{e_\kappa} \nabla (e_\kappa \psi_{\xi^\varepsilon_\theta}) + c_{\xi^\varepsilon_\theta}) \cdot \overline{A^{-1/2} (\overline{e_\kappa} \nabla (e_\kappa \psi_{R^\varepsilon_\theta}) + c_{R^\varepsilon_\theta})}.\end{aligned}$$

The second expression in the right hand side of last equation vanishes, because it is the equation (6.34) tested with $A^{-1/2}\overline{e_\kappa}\nabla(e_\kappa \Phi_{R^\varepsilon_\theta})$. The third expression in the right hand side vanishes as well. In fact, it is the equation (6.34) tested with the element $A^{-1/2}(\overline{e_\kappa}\nabla(e_\kappa \psi_{R^\varepsilon_\theta}) + c_{R^\varepsilon_\theta})$, which vanishes taking into account (6.20).

Hence, we can write equation (6.32) as

$$\begin{aligned}\int_Q &\widetilde{A} \operatorname{curl}(e_\kappa A^{1/2} R^\varepsilon_\theta) \cdot \overline{\operatorname{curl}(e_\kappa A^{1/2} R^\varepsilon_\theta)} + \varepsilon^2 \int_Q \left| A^{-1/2} (\overline{e_\kappa} \nabla (e_\kappa \psi_{R^\varepsilon_\theta}) + c_{R^\varepsilon_\theta}) \right|^2 \\ &+ \varepsilon^2 \int_Q \left| A^{-1/2} \overline{e_\kappa} \nabla (e_\kappa \Phi_{R^\varepsilon_\theta}) \right|^2 = - \int_Q \Big(A^{1/2} \big(G + i\theta \times \widetilde{A}(i\theta \times d^\varepsilon_\theta) \\ &+ i\theta \times \widetilde{A} \operatorname{curl}(A^{1/2} N(i\theta \times d^\varepsilon_\theta)) \big) + A^{-1/2} (\overline{e_\kappa} \nabla (e_\kappa \Psi_\kappa) + I) d^\varepsilon_\theta \Big) \cdot \overline{\widetilde{R^\varepsilon_\theta}} - \overline{\langle \mathcal{H}^\varepsilon_\theta, \widetilde{\xi^\varepsilon_\theta} \rangle}.\end{aligned} \tag{6.37}$$

Lemma 6.3.2. *The last term in the right hand side of* (6.37) *is bounded uniformly in* ε *and* θ*:*

$$|\langle \mathcal{H}^\varepsilon_\theta, \widetilde{\xi^\varepsilon_\theta} \rangle| \leq C \|G\|_{L^2(Q,d\mu)}, \quad C > 0$$

Proof. By the definition (6.25) for $\mathcal{H}^\varepsilon_\theta$, we have

$$\begin{aligned}\langle \mathcal{H}^\varepsilon_\theta, \widetilde{\xi^\varepsilon_\theta} \rangle = &- \int_Q \Big(A^{1/2} \big(G + i\theta \times \widetilde{A}(i\theta \times d^\varepsilon_\theta) + i\theta \times \widetilde{A} \operatorname{curl}(A^{1/2} N(i\theta \times d^\varepsilon_\theta)) \\ &+ \varepsilon i\theta \times \widetilde{A}(i\theta \times A^{1/2} N(i\theta \times d^\varepsilon_\theta) \big) + A^{-1/2} \overline{e_\kappa} (\nabla (e_\kappa \Psi_\kappa) + I) d^\varepsilon_\theta \Big) \cdot \overline{\widetilde{\xi^\varepsilon_\theta}} \\ &- \int_Q \widetilde{A} i\theta \times (A^{1/2} N(i\theta \times d^\varepsilon_\theta)) \cdot \overline{\operatorname{curl}(A^{1/2} \widetilde{\xi^\varepsilon_\theta})}.\end{aligned}$$

Using the following identity for $\phi_n \in C^1_\#$ (*cf.* with (6.30))

$$\operatorname{curl}(A^{1/2} \phi_n) = i\kappa \times A^{1/2} \phi_n + \overline{e_\kappa} \operatorname{curl}(e_\kappa A^{1/2} \phi_n),$$

we obtain

$$\langle \mathcal{H}_\theta^\varepsilon, \widetilde{\xi_\theta^\varepsilon} \rangle = -\int_Q \biggl(A^{1/2}\bigl(G + i\theta \times \widetilde{A}(i\theta \times d_\theta^\varepsilon) + i\theta \times \widetilde{A}\,\mathrm{curl}(A^{1/2}N(i\theta \times d_\theta^\varepsilon))\bigr)$$
$$+ A^{-1/2}\overline{e_\kappa}(\nabla(e_\kappa \Psi_\kappa) + I)d_\theta^\varepsilon \biggr) \cdot \overline{\widetilde{\xi_\theta^\varepsilon}} - \int_Q e_\kappa \widetilde{A} i\theta \times (A^{1/2}N(i\theta \times d_\theta^\varepsilon)) \cdot \overline{\mathrm{curl}(e_\kappa A^{1/2}\widetilde{\xi_\theta^\varepsilon})}.$$

Applying the decomposition (5.19) to ξ_θ^ε, we note that

$$\int_Q e_\kappa \widetilde{A} i\theta \times (A^{1/2}N(i\theta \times d_\theta^\varepsilon)) \cdot \overline{\mathrm{curl}(e_\kappa A^{1/2}\widetilde{\xi_\theta^\varepsilon})} = \int_Q e_\kappa \widetilde{A} i\theta \times (A^{1/2}N(i\theta \times d_\theta^\varepsilon)) \cdot \overline{\mathrm{curl}(e_\kappa A^{1/2}\xi_\theta^\varepsilon)},$$

since

$$\int_Q e_\kappa \widetilde{A} i\theta \times (A^{1/2}N(i\theta \times d_\theta^\varepsilon)) \cdot \overline{\mathrm{curl}(e_\kappa c_{\xi_\theta^\varepsilon})} = 0.$$

The last equality follows from (6.20), noting that $\mathrm{curl}(e_\kappa c_{\xi_\theta^\varepsilon}) = e_\kappa(i\kappa \times c_{\xi_\theta^\varepsilon})$. Now, using the Hölder inequality, the Poincaré-type inequality (5.22) for ξ_θ^ε and taking into account the estimate (6.35), the required statement holds. $\square$

Combining the Lemma 6.3.2, the Hölder inequality and the Poincaré-type inequality (5.22) for R_θ^ε in (6.37) we obtain the following uniform bound

$$\bigl\| \mathrm{curl}(e_\kappa A^{1/2} R_\theta^\varepsilon) \bigr\|_{L^2(Q,d\mu)} \le \varepsilon C \|G\|_{L^2(Q,d\mu)}. \tag{6.38}$$

The estimate (6.38) combined with (5.22), implies (6.28). Furthermore the same estimate, Lemma 6.3.2 and equation (6.37) imply (6.29). $\square$

Corollary 6.3.3. *There exists a constant $C > 0$ such that the following estimate holds uniformly in ε, θ and G:*

$$\|U_\theta^\varepsilon - A^{-1/2}(\overline{e_\kappa}(\nabla e_\kappa \Psi_\kappa) + I)d_\theta^\varepsilon\|_{L^2(Q,d\mu)} \le \varepsilon C \|G\|_{L^2(Q,d\mu)}.$$

6.3.1 Conclusion of the convergence estimate

Proposition 6.3.4. *There exists $C > 0$ such that the function z_θ^ε defined in (6.17) satisfies the estimate*

$$\|z_\theta^\varepsilon\|_{L^2(Q,d\mu)} \le C\varepsilon \|G\|_{L^2(Q,d\mu)}. \tag{6.39}$$

Proof. The vector function $z_\theta^\varepsilon \in H^1_{\mathrm{curl}\,A^{1/2}}(Q, d\mu)$ solves

$$\varepsilon^{-2} A^{1/2}\overline{e_\kappa}\,\mathrm{curl}\,\widetilde{A}\,\mathrm{curl}\,e_\kappa A^{1/2} z_\theta^\varepsilon + z_\theta^\varepsilon = -\varepsilon^2 \widetilde{R_\theta^\varepsilon} - \varepsilon N(i\theta \times d_\theta^\varepsilon). \tag{6.40}$$

Using z^ε_θ as a test function in the equation (6.40), on has

$$\varepsilon^{-2}\int_Q \widetilde{A}\,\mathrm{curl}(e_\kappa A^{1/2} z^\varepsilon_\theta)\cdot\overline{\mathrm{curl}(e_\kappa A^{1/2} z^\varepsilon_\theta)}+\int_Q |z^\varepsilon_\theta|^2=-\varepsilon\int_Q N(i\theta\times d^\varepsilon_\theta)\cdot\overline{z^\varepsilon_\theta}-\varepsilon^2\int_Q \widetilde{R^\varepsilon_\theta}\cdot\overline{z^\varepsilon_\theta}. \tag{6.41}$$

Using the Hölder inequality, the Poincaré-type inequality (5.22) and the definition (6.13) for d^ε_θ, one has the (6.39). □

Note that applying the Hölder inequality, and the estimates (6.38), (6.39) to the right hand side of (6.41), one has

$$\|\,\mathrm{curl}(e_\kappa A^{1/2} z^\varepsilon_\theta)\|_{L^2(Q,d\mu)}\le C\varepsilon^2\|G\|_{L^2(Q,d\mu)}. \tag{6.42}$$

Proposition 6.3.4 and Corollary 6.3.3 imply (6.12), in fact

$$\begin{aligned}&\big\|D^\varepsilon_\theta-A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta})+I\big)d^\varepsilon_\theta\big\|_{L^2(Q,d\mu)}\\&\le\|U^\varepsilon_\theta-A^{-1/2}\big(\overline{e_{\varepsilon\theta}}\nabla(e_{\varepsilon\theta}\Psi_{\varepsilon\theta})+I\big)d^\varepsilon_\theta\|_{L^2(Q,d\mu)}+\|z^\varepsilon_\theta\|_{L^2(Q,d\mu)},\end{aligned}$$

hence the proof of Theorem 6.2.3 is concluded.

6.3.2 Estimate for magnetic field and magnetic induction

To obtain the estimates for the magnetic field and induction, we start from the equation (6.6). The transformed problem can be written as

$$\begin{cases}\varepsilon^{-1}\overline{e_\kappa}\,\mathrm{curl}(e_\kappa A^{1/2}D^\varepsilon_\theta)+B^\varepsilon_\theta=0,\\ \varepsilon^{-1}A^{1/2}\overline{e_\kappa}\,\mathrm{curl}(e_\kappa\widetilde{A}B^\varepsilon_\theta)-D^\varepsilon_\theta=A^{1/2}G.\end{cases} \tag{6.43}$$

where $\overline{e_\kappa}\,\mathrm{div}(e_\kappa A^{-1/2}D^\varepsilon_\theta)=0$, and $B^\varepsilon_\theta:=\overline{e_\kappa}\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon\mathcal{B}_\varepsilon$ is the transformed magnetic induction such that $\overline{e_\kappa}\,\mathrm{div}(e_\kappa B^\varepsilon_\theta)=0$. In this case the transformed magnetic field $H^\varepsilon_\theta=\widetilde{A}\,B^\varepsilon_\theta$.

To find the approximation for B^ε_θ we use (6.17) in the first line of the system (6.43). Hence, we have that

$$\begin{aligned}B^\varepsilon_\theta=&\varepsilon^{-1}\overline{e_\kappa}\,\mathrm{curl}\,e_\kappa\big((\overline{e_\kappa}\nabla(e_\kappa\Psi_\kappa)+I)d^\varepsilon_\theta+\varepsilon A^{1/2}N(i\theta\times d^\varepsilon_\theta)+\varepsilon^2A^{1/2}R^\varepsilon_\theta+A^{1/2}z^\varepsilon_\theta\big)\\=&(\mathrm{curl}\,A^{1/2}N+I)(i\theta\times d^\varepsilon_\theta)+\varepsilon\big(i\theta\times(A^{1/2}N(i\theta\times d^\varepsilon_\theta))+\overline{e_\kappa}\,\mathrm{curl}(e_\kappa A^{1/2}R^\varepsilon_\theta)\big)\\&+\varepsilon^{-1}\overline{e_\kappa}\,\mathrm{curl}(e_\kappa A^{1/2}z^\varepsilon_\theta).\end{aligned}$$

Here d^ε_θ is the solution of (6.13), N is defined in (6.19), R^ε_θ solves (6.23) and z^ε_θ solves (6.40). As a consequence of (6.38) and (6.42) we can state the following result.

Theorem 6.3.5. *Under the assumptions on the measure μ and the coefficients A, $\widetilde{A}$ stated in Theorem 6.2.3, there exists $C>0$ independent of θ, ε and G such that the* *following estimates hold for the transformed magnetic induction B_θ^ε and the transformed magnetic field $H_\theta^\varepsilon := \widetilde{A} B_\theta^\varepsilon$*

$$\|B_\theta^\varepsilon - (\operatorname{curl}(A^{1/2}N) + I)(i\theta \times d_\theta^\varepsilon)\|_{L^2(Q,d\mu)} \leq \varepsilon C \|G\|_{L^2(Q,d\mu)} \tag{6.44}$$
$$\|H_\theta^\varepsilon - \widetilde{A}(\operatorname{curl}(A^{1/2}N) + I)(i\theta \times d_\theta^\varepsilon)\|_{L^2(Q,d\mu)} \leq \varepsilon C \|G\|_{L^2(Q,d\mu)} \tag{6.45}$$

Applying back the Floquet transform on (6.44) on obtains the following norm-resolvent estimates on the whole space setting for $\mathcal{B}_\varepsilon$ solution of (6.2). (Analogously starting from (6.45) on obtain estimates for $\mathcal{H}_\varepsilon := \widetilde{A}\mathcal{B}_\varepsilon$.)

Corollary 6.3.6. *Let $g^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ and denote $g_\theta^\varepsilon := \overline{e_\kappa}\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon g^\varepsilon$ so that*

$$\int_Q g_\theta^\varepsilon d\mu = \widehat{g}(\theta, \varepsilon), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \widehat{g}(\theta,\varepsilon) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} g^\varepsilon \overline{e_\theta}\, d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.$$

There exists a constant $C > 0$ such that the following estimate holds for $\mathcal{B}_\varepsilon$ solution of (6.2)

$$\left\| \mathcal{B}_\varepsilon - (\operatorname{curl}(A^{1/2}N)(\cdot/\varepsilon) + I) \operatorname{curl}\left((2\pi)^{-3/2} \int_{\mathbb{R}^3} \big(\widehat{A}_{\varepsilon\theta}^{\hom}\big)^{-1} \big(\widetilde{\mathfrak{A}}_\theta^{\hom} + I\big)^{-1} \widehat{g}(\theta,\varepsilon) e_\theta d\theta \right) \right\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)} \leq C\varepsilon \|g^\varepsilon\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)}, \tag{6.46}$$

$\forall \varepsilon > 0$. *Here $\widetilde{\mathfrak{A}}_\theta^{\hom}$ is the matrix valued quadratic form given by* (6.13), *and N is defined in* (6.19).

As discussed in Section 5.3.2, the estimate (6.46) allows us to approximate the magnetic induction $\mathcal{B}_\varepsilon$ with

$$(\operatorname{curl}(A^{1/2}N) + I) \operatorname{curl}\left((2\pi)^{-3/2} \int_{\mathbb{R}^3} \big(\widehat{A}_{\varepsilon\theta}^{\hom}\big)^{-1} \big(\widetilde{\mathfrak{A}}_\theta^{\hom} + I\big)^{-1} \widehat{g}(\theta,\varepsilon) e_\theta d\theta \right), \tag{6.47}$$

that is the correct replacement for the homogenised solution operator for the full Maxwell system. The structure of (6.47) is similar to the one obtained in the estimate (5.55) in the whole space setting for the magnetic induction in Chapter 5, for the Maxwell system with relative magnetic permeability set to unity.

The differences between (5.56) and (6.47) are that in (6.47) appears the multiplication by the divergence-free element $(\operatorname{curl}(A^{1/2}N) + I)$. Furthermore, in the operator $\widetilde{\mathfrak{A}}_\theta^{\hom}$ there is the matrix $\widetilde{A}^{\hom}$ which is constant with respect to $\varepsilon\theta$ and does not influence the structure of the solution operator.

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